

# Exact computation of one-loop correction to energy of pulsating strings in $AdS_5 \times S^5$

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## Abstract

In the present paper, which is a sequel to arXiv:1001.4018, we compute the one-loop correction to the energy of pulsating string solutions in  $AdS_5 \times S^5$ . We show that, as for rigid spinning string elliptic solutions, the fluctuation operators for pulsating solutions can be also put into the single-gap Lamé form. A novel aspect of pulsating solutions is that the one-loop correction to their energy is expressed in terms of the stability angles of the quadratic fluctuation operators. We explicitly study the “short string” limit of the corresponding one-loop energies, demonstrating a certain universality of the form of the energy of “small” semiclassical strings. Our results may help to shed light on the structure of strong-coupling expansion of anomalous dimensions of dual gauge theory operators.

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## 1 Introduction

$AdS_5 \times S^5$  string energies (or planar  $\mathcal{N}=4$  SYM anomalous dimensions) are, in general, complicated functions of string tension (or ‘t Hooft coupling) and various charges. They should be described by the integrability-based Thermodynamic Bethe Ansatz (see, e.g., [1] for a review) but detailed patterns of their behaviour with coupling and charges are still poorly understood.

The semiclassical string expansion applies in a particular limit when charges scale as string tension  $\frac{\sqrt{\lambda}}{2\pi} \gg 1$ . It was argued [2] that in this limit the TBA prediction matches exactly the 1-loop string correction to the energies. However, the details of the correspondence between the Bethe ansatz (or algebraic curve) description of the semiclassical solutions and their direct 2d string sigma model description remain to be clarified for various non-trivial types of solutions. Also, following [3, 4] one may hope to shed light on the structure of anomalous dimensions of “short” operators by studying the “short string” limit – the limit of small values of the semiclassical parameters. Assuming that this limit commutes with the large  $\sqrt{\lambda}$  limit one could then interpolate the result to small (fixed) values of the quantum string charges.

Apart from the case of the rational rigid string solutions for which the fluctuation Lagrangian has constant coefficients [5], the direct 2d quantum field theory computation of the 1-loop correction to string energies is difficult. To compute the one-loop energy one needs to find the spectrum of mixed-mode fluctuation operators which are second order matrix 2d differential operators with coordinate-dependent coefficients. As was explained in the previous paper of three of us [6], for the next to the simplest case of elliptic solutions with the folded spinning string being the basic example, one can compute the corresponding determinants using the special Lamé form of the fluctuation operators.

The present paper is a natural sequel to [6] where we treat other cases of similar elliptic solutions – pulsating string solutions in  $AdS_5$  and  $S^5$ . A novel aspect of pulsating solutions – which are time-dependent rather than rigid stationary as in previous spinning string case – is that the 1-loop correction to their energy is determined in a more complicated way than just by summing characteristic frequencies. As we shall discuss below, in this case one needs to follow the general semiclassical method of quantization of time-periodic solitons [7, 8] expressing the correction to the energy in terms of the stability angles of the fluctuation operators.

The structure of this paper is as follows. In section 2 we discuss a simple pulsating string solution on  $S^2$ . We present the bosonic and fermionic quadratic fluctuation operators and show that these can be written as 1-d differential operators with single-gap Lamé potentials.

In section 3 we repeat the same for a pulsating string in  $AdS_3$ . In section 4 we review the general semiclassical quantization approach focussing on the case of time-dependent potentials, where one needs to use the stability angles to compute one-loop correction to the energy.

In section 5 we use the data of the previous sections to write down the complete one-loop correction to the energy of pulsating string solutions. We then derive the explicit expansion of the energy in the short string limit.

In section 6 we return to a rigid spinning string case of a type considered in [6] – the folded string

in  $\mathbb{R} \times S^2$ . We find the exact one-loop correction to its energy and expand it explicitly in the short string limit.

Finally, in section 7 we present our conclusions and compare the short-string one-loop energies for all the four cases of the folded and pulsating strings in  $AdS_5 \times S^5$  studied in [6] and here.

Our notations and some details of the computation of fluctuation operators are presented in appendices A–D. Appendix E supplements the discussion in section 4 presenting a heuristic derivation of the one-loop expression for the energy in terms of the stability angles. In Appendix F we discuss an alternative antiperiodic choice for the fermionic boundary condition mentioned in section 7.

## 2 Pulsating string in $\mathbb{R} \times S^2$

In this section we shall start with the classical string background representing pulsating string in  $\mathbb{R} \times S^2$  and then consider the second order 2d operators governing the spectrum of small fluctuations near this solution. We shall then demonstrate that as in the case of another elliptic solution – folded string in  $AdS_3$  considered in [6] these operators can be put in the Lamé form, and thus their spectrum and eventually the one-loop correction to the string energy can be computed exactly.

### 2.1 Classical solution

The pulsating string solution in  $\mathbb{R} \times S^2$ , which is a generalization of a circular pulsating string on a plane was considered in [9] and in [10] starting with the Nambu action in the static gauge. Here we review this solution in the conformal gauge following [11]. Let us start with the following ansatz for the bosonic string coordinates in  $\mathbb{R} \times S^2$  ( $m = 1, 2, \dots$ )

$$t = \kappa \tau, \quad \psi = \psi(\tau), \quad \phi = m \sigma, \quad ds^2 = -dt^2 + d\psi^2 + \sin^2 \psi d\phi^2. \quad (2.1)$$

The equation of motion and the conformal gauge constraint (which implies the former for  $\dot{\psi} \neq 0$ ) are

$$\ddot{\psi} + m^2 \sin \psi \cos \psi = 0, \quad \dot{\psi}^2 + m^2 \sin^2 \psi = \kappa^2. \quad (2.2)$$

The solution with  $\psi(0) = 0$  can be written in terms of the Jacobi elliptic function [16]

$$\sin \psi(\tau) = \frac{\kappa}{m} \operatorname{sn}\left(m \tau \mid \frac{\kappa^2}{m^2}\right), \quad |\sin \psi| \leq \sin \psi_0 = \frac{\kappa}{m}. \quad (2.3)$$

To have a time-periodic solution we need to assume  $\kappa < m$ . The induced metric and its curvature are

$$ds^2 = m^2 \sin^2 \psi \eta_{ab}, \quad R^{(2)} = \frac{2}{m^2 \sin^2 \psi} \left( m^2 \sin^2 \psi - \frac{\kappa^2}{\sin^2 \psi} \right). \quad (2.4)$$

The energy and the oscillation number  $N = \frac{\sqrt{\lambda}}{2\pi} \oint d\psi \dot{\psi}$  (the adiabatic invariant associated to  $\psi$ ) are<sup>1</sup>

$$\mathcal{E}_0 = \frac{E}{\sqrt{\lambda}} = \kappa, \quad (2.5)$$

$$\mathcal{N} = \frac{N}{\sqrt{\lambda}} = \int_0^{2\pi} \frac{d\psi}{2\pi} \sqrt{\kappa^2 - m^2 \sin^2 \psi} = \int_0^{4\mathbb{K}(\frac{\kappa^2}{m^2})} \frac{d\tau}{2\pi} \left( \kappa^2 - m^2 \sin^2 \psi \right). \quad (2.6)$$

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<sup>1</sup>  $\frac{\sqrt{\lambda}}{2\pi}$  is string tension. We follow the same notation for elliptic functions as in [6].

A short calculation gives

$$\mathcal{N} = \frac{2m}{\pi} \left[ \left( \frac{\kappa^2}{m^2} - 1 \right) \mathbb{K} \left( \frac{\kappa^2}{m^2} \right) + \mathbb{E} \left( \frac{\kappa^2}{m^2} \right) \right] . \quad (2.7)$$

where  $\mathbb{K}$  and  $\mathbb{E}$  are the usual elliptic functions [16, 6]. The condition  $\kappa < m$  gives an upper bound for  $\mathcal{N}$ , i.e. here, like for the folded string in  $\mathbb{R} \times S^2$  (but in contrast to the folded string in  $AdS_3$ ) one cannot take the large  $\mathcal{N}$  limit.

The expansion of  $\mathcal{N}$  for small  $\kappa$  gives

$$\mathcal{N} = \frac{\kappa^2}{2m} + \frac{\kappa^4}{16m^3} + \frac{3\kappa^6}{128m^5} + \dots \quad (2.8)$$

Thus the short string or small oscillation number ( $\mathcal{N} \rightarrow 0$ ) expansion of the classical energy is

$$\mathcal{E}_0(\mathcal{N}) = \sqrt{2m\mathcal{N}} \left( 1 - \frac{\mathcal{N}}{8m} - \frac{5\mathcal{N}^2}{128m^2} + \dots \right). \quad (2.9)$$

## 2.2 Quadratic fluctuation Lagrangian

In order to compute the 1-loop correction to energy (2.9) we need to find the operators of quadratic fluctuations. The derivation is standard with details presented in Appendix B (we follow Appendix A of [12]).

The bosonic fluctuation operators can be found directly in conformal gauge where we find two mixed modes. They can be decoupled by solving the Virasoro constraints with the resulting fluctuation action (with 2 “longitudinal” massless modes omitted) being equivalent to the one that can be found directly using the static gauge as in [6].

The conformal gauge fluctuations in  $AdS_5$  directions are represented by a free massless “ghost” field plus four free massive fields with mass  $\kappa$  (here  $k = 1, 2, 3, 4$ ;  $\partial_a \partial^a = -\partial_\tau^2 + \partial_\sigma^2$ )

$$L_{AdS}^{(2)} = -\frac{1}{2}(\dot{\beta}^2 - \beta'^2) + \frac{1}{2}(\dot{y}_k^2 - y_k'^2 - \kappa^2 y_k y_k) . \quad (2.10)$$

The Lagrangian for the five  $S^5$  fluctuations  $(\xi, \eta, z_1, z_2, z_3)$  is

$$\begin{aligned} L_S^{(2)} = & \frac{1}{2}(\dot{\xi}^2 - \xi'^2 - M_\xi^2 \xi^2) + \frac{1}{2}(\dot{\eta}^2 - \eta'^2 - M_\eta^2 \eta^2) + m \cos \psi (\xi \eta' - \xi' \eta) \\ & + \frac{1}{2}(\dot{z}_i^2 - z_i'^2 - M^2 z_i^2) , \end{aligned} \quad (2.11)$$

where the background-dependent masses are

$$M^2 = \kappa^2 - 2m^2 \sin^2 \psi, \quad M_\xi^2 = \kappa^2 + m^2 \cos(2\psi) , \quad M_\eta^2 = m^2 \cos(2\psi) . \quad (2.12)$$

Solving the Virasoro constraints one can show that the coupled system  $(\xi, \eta)$  is equivalent to a decoupled system of one massless mode and of the massive mode with the Lagrangian

$$L = \frac{1}{2}(\dot{g}^2 - g'^2 - \tilde{M}^2 g^2) , \quad \tilde{M}^2 = \kappa^2 \left( 1 - \frac{2}{\sin^2 \psi} \right) . \quad (2.13)$$

This is the same fluctuation Lagrangian (with the massless modes omitted) as found by starting with the Nambu action and imposing the static gauge on the fluctuations (see Appendix A). An equivalent fluctuation action follows also from the Pohlmeyer reduction approach [13].

The fermionic fluctuation Lagrangian is found as, e.g., in [14, 6]. In the standard  $\theta^1 = \theta^2$  kappa symmetry gauge it is (cf. Appendix A)

$$\mathcal{L}_F = -2i\bar{\vartheta}\left(-\rho^a D_a - \frac{i}{2}\varepsilon^{ab}\rho_a\Gamma_*\rho_b\right)\vartheta, \quad (2.14)$$

leading to the following expression for the fermionic fluctuation operator (see Appendix B)

$$D_F = \Gamma_0\partial_\tau - \Gamma_9\partial_\sigma + \Gamma_{079}\dot{\psi}. \quad (2.15)$$

Since we are interested in its eigenvalues and determinant, we can take the square of the simpler operator

$$\tilde{D}_F \equiv \Gamma_{09} D_F = \Gamma_9\partial_\tau - \Gamma_0\partial_\sigma - \Gamma_7\dot{\psi}. \quad (2.16)$$

Diagonalizing  $\Gamma_{97}$  (i.e. replacing it by  $\pm i$ ) we get the following second order fermionic operator

$$\tilde{D}_{F\pm}^2 = \partial_\tau^2 - \partial_\sigma^2 + M_\pm^2, \quad M_\pm^2 = \dot{\psi}^2 \pm i\ddot{\psi}. \quad (2.17)$$

A simple check on the resulting fluctuation Lagrangian is provided by demonstrating the UV finiteness of the 1-loop partition function. In conformal gauge that requires showing that the sum of the effective mass-squared terms for bosons equals that for the fermions.<sup>2</sup> We find that the sum of the physical 4+4 bosonic and 4+4 fermionic effective mass squared terms<sup>3</sup>

$$\begin{aligned} AdS &: 4 \times \kappa^2, \\ S^5 &: 3 \times (\kappa^2 - 2m^2 \sin^2 \psi), \\ &\quad 1 \times (m^2 \cos(2\psi) - m^2 \cos^2 \psi), \\ &\quad 1 \times (\kappa^2 + m^2 \cos(2\psi) - m^2 \cos^2 \psi), \\ F &: -8 \times (\kappa^2 - m^2 \sin^2 \psi) \end{aligned} \quad (2.18)$$

indeed sums to zero. In the static gauge we get

$$\begin{aligned} AdS &: 4 \times \kappa^2, \\ S^5 &: 3 \times (\kappa^2 - 2m^2 \sin^2 \psi), \\ &\quad 1 \times \kappa^2 \left(1 - \frac{2}{\sin^2 \psi}\right), \\ F &: -8 \times (\kappa^2 - m^2 \sin^2 \psi) \end{aligned} \quad (2.19)$$

and the sum is

$$2m^2 \sin^2 \psi - 2\frac{\kappa^2}{\sin^2 \psi} = \sqrt{-g} R^{(2)}, \quad (2.20)$$

where  $\sqrt{-g} R^{(2)}$  is proportional to the Euler density of the induced metric as expected on general grounds [15]. As discussed in [6], integrated over the 2-space, this is proportional to the Euler number which vanishes for the cylinder topology under discussion.

<sup>2</sup>The contribution of the two mixed fluctuations can be found by rewriting the corresponding terms as  $A'^2 + B'^2 + \mu AB' = (A' - \frac{\mu}{4}B)^2 + (B' + \frac{\mu}{4}A)^2 - \frac{\mu^2}{16}(A^2 + B^2)$  and observing that the “connection” terms do not produce UV divergences.

<sup>3</sup>For this counting argument we may ignore the  $\pm i\ddot{\psi}$  terms in the fermionic masses in (2.17), as they sum up to zero.

### 2.3 Remarks on single-gap Lamé operator

We will show in the next section that each of the above quadratic fluctuation operators can be transformed into the “single-gap Lamé” form

$$\left[ -\partial_x^2 + 2k^2 \operatorname{sn}^2(x|k^2) \right] f(x) = \Lambda f(x) \quad , \quad (2.21)$$

for suitable choices of the coordinate  $x$  and the elliptic parameter  $k^2$ . This fact is significant because equation (2.21) has simple solutions and properties, which we review briefly here (see also [16, 6]). The two independent Bloch solutions of (2.21) are

$$f_{\pm}(x) = \frac{H(x \pm \alpha)}{\Theta(x)} e^{\mp x Z(\alpha)} \quad , \quad (2.22)$$

where  $H, \Theta, Z$  are the Jacobi Eta, Theta and Zeta functions [16], and the spectral parameter  $\alpha = \alpha(\Lambda)$  is related to the eigenvalue  $\Lambda$  by the transcendental equation:

$$\operatorname{sn}(\alpha|k^2) = \sqrt{\frac{1+k^2-\Lambda}{k^2}} \quad . \quad (2.23)$$

Using the periodicity properties of the Jacobi functions we see that the Bloch solutions  $f_{\pm}(x)$  acquire a phase under a shift through one period  $2\mathbb{K}$ :

$$f_{\pm}(x + 2\mathbb{K}) = -f_{\pm}(x) e^{\mp 2\mathbb{K} Z(\alpha)} \equiv f_{\pm}(x) e^{2i\mathbb{K} p(\alpha)} \quad . \quad (2.24)$$

This defines the quasi-momentum as

$$p(\Lambda) = i Z(\alpha|k^2) + \frac{\pi}{2\mathbb{K}} \quad . \quad (2.25)$$

As explained in [6], knowing an explicit expression for the quasi-momentum implies that we can write an explicit expression for the corresponding determinant of the fluctuation operator. We will return to this in sections 4 and 5.

### 2.4 Lamé form of fluctuation operators

Having motivated the significance of the single-gap Lamé form of the fluctuation operators, we will now present their explicit form for each of the decoupled (static gauge) fluctuation operators in the pulsating string case. Since the fluctuation potentials are independent of  $\sigma$  for the pulsating string solutions, we may use the Fourier decomposition of the  $\sigma$  dependence,  $X(\tau, \sigma) = X(\tau) e^{in\sigma}$ , so that  $-\partial_\tau^2 + \partial_\sigma^2 + M^2(\tau) \rightarrow -\partial_\tau^2 + M^2(\tau) - n^2$ . Depending on the form of the mass term (i.e. potential)  $M^2(\tau)$ , we find three types of Lamé operators, which we discuss in turn.

#### Type I operator

The operator associated to the three  $S^5$  modes  $z_i$  in (2.11) with mass  $M^2 = \kappa^2 - 2m^2 \sin^2 \psi$  is

$$\mathcal{O}_I = -\partial_\tau^2 + 2m^2 \sin^2 \psi - \kappa^2 - n^2 \quad . \quad (2.26)$$

Taking into account the specific form of the solution  $\psi(\tau)$  in (2.3), it can be written as

$$\mathcal{O}_I = m^2 \left[ -\partial_x^2 + 2k^2 \operatorname{sn}^2(x|k^2) - \Lambda \right] \quad , \quad (2.27)$$

$$x = m\tau, \quad k^2 = \frac{\kappa^2}{m^2}, \quad \Lambda = \frac{\kappa^2 + n^2}{m^2} \quad , \quad (2.28)$$

which is of the single-gap Lamé form in (2.21). The classical stability region is the set of  $\Lambda$  for which the quasi-momentum is real and the solution of the differential equation is quasi-periodic. Outside this region, the solution is unbounded and is not acceptable. The Lamé operator in (2.27) is called “single-gap” because there are just two allowed bands separated by a single gap:

$$\Lambda \in [k^2, 1] \cup [k^2 + 1, +\infty] . \quad (2.29)$$

Assuming  $\kappa < m$  this gives

$$n \in [0, \sqrt{m^2 - \kappa^2}] \cup [m, +\infty] . \quad (2.30)$$

Notice that for  $\kappa \leq \sqrt{2m-1}$  the above range covers all integers  $n$ . If this condition is not satisfied, there are certain values of  $n$  which give rise to unstable fluctuations.<sup>4</sup>

### Type II operator

Next, consider the  $S^5$  mode in (2.13) with mass  $\tilde{M}^2 = \kappa^2(1 - \frac{2}{\sin^2 \psi})$ , i.e. with the associated operator

$$\mathcal{O}_{II} = -\partial_\tau^2 + \frac{2\kappa^2}{\sin^2 \psi} - \kappa^2 - n^2 . \quad (2.31)$$

After using (2.3) and definitions in (2.28) we get

$$\mathcal{O}_{II} = m^2 \left[ -\partial_x^2 + 2 \operatorname{ns}^2(x | k^2) - \Lambda \right] . \quad (2.32)$$

Taking into account the identity,  $\operatorname{ns}(z | k^2) = k \operatorname{sn}(z + i \mathbb{K}' | k^2)$ , we can write<sup>5</sup>

$$\mathcal{O}_{II} = m^2 \left[ -\partial_x^2 + 2 k^2 \operatorname{sn}^2(x | k^2) - \Lambda \right] , \quad (2.33)$$

$$x \equiv m \tau + i \mathbb{K}' , \quad k = \frac{\kappa}{m} , \quad \Lambda = \frac{\kappa^2 + n^2}{m^2} , \quad (2.34)$$

which is again of the single-gap Lamé form in (2.21).

### Type III operator

The fermion fluctuation operator in (2.17) with the mass  $M_\pm^2 = \dot{\psi}^2 \pm i \ddot{\psi}$  leads to

$$\mathcal{O}_{III}^\pm = -\partial_\tau^2 - \dot{\psi}^2 \mp i \ddot{\psi} - n^2 . \quad (2.35)$$

Using the explicit form of  $\psi(\tau)$  in (2.3) and (2.28) we get

$$\mathcal{O}_{III}^\pm = m^2 \left[ -\partial_x^2 - k^2 \operatorname{cn}^2(x | k^2) \mp i k \operatorname{sn}(x | k^2) \operatorname{dn}(x | k^2) - \frac{n^2}{m^2} \right] , \quad (2.36)$$

$$x \equiv m \tau , \quad k = \frac{\kappa}{m} , \quad \Lambda = \frac{n^2}{m^2} . \quad (2.37)$$

<sup>4</sup>Let us mention that a hybrid string solution with two spins in  $AdS_5$  and pulsating in  $S^5$  was considered before in [17]. The numerical analysis in [17] showed that generally pulsation improves the stability of a spinning string.

<sup>5</sup>We use the standard notation  $\mathbb{K}'(k^2) \equiv \mathbb{K}(1 - k^2)$ .



This operator is non-hermitian, but is PT-symmetric [18] and has a real spectrum.<sup>6</sup> Moreover, while it does not look like the standard single-gap Lamé operator, it can be transformed into this form by a combination of rescaling of  $x$  and a Gauss transformation of the elliptic parameter  $k^2$ ,

$$\mathcal{O}_{III}^{\pm} = \bar{m}_{\pm}^2 \left[ -\partial_x^2 + 2 \bar{k}_{\pm}^2 \operatorname{sn}^2(\bar{x} | \bar{k}_{\pm}^2) - \Lambda \right] , \quad (2.38)$$

$$\bar{x} \equiv \bar{m}_{\pm} \tau + \frac{1}{2} \mathbb{K}(\bar{k}_{\pm}^2) , \quad \bar{m}_{\pm} = \frac{m}{2} \left( \sqrt{1 - \frac{\kappa^2}{m^2}} \pm i \frac{\kappa}{m} \right) , \quad (2.39)$$

$$\bar{k}_{\pm}^2 = \pm 4 \frac{\frac{i\kappa}{m} \sqrt{1 - \frac{\kappa^2}{m^2}}}{\left( \sqrt{1 - \frac{\kappa^2}{m^2}} \pm \frac{i\kappa}{m} \right)^2} , \quad \Lambda = \frac{n^2}{\bar{m}_{\pm}^2 + \bar{k}_{\pm}^2} . \quad (2.40)$$

Thus we again find a fluctuation operator of the single-gap Lamé form in (2.21).

### 3 Pulsating string in $AdS_3$

Our aim here will be to repeat the discussion of the previous section in the case of the pulsating string solution in  $AdS_3$  discussed in [10, 19].

#### 3.1 Classical solution

Using the standard parametrization of the  $AdS_5$  metric

$$ds_{AdS_5}^2 = d\rho^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho (d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2) , \quad (3.1)$$

let us look for a string solution in conformal gauge assuming

$$t = t(\tau), \quad \rho = \rho(\tau), \quad \theta = 0, \quad \phi_1 = m\sigma, \quad \phi_2 = 0 . \quad (3.2)$$

The non-trivial conformal gauge constraint and the two equations of motion are

$$\dot{\rho}^2 - \cosh^2 \rho \dot{t}^2 + m^2 \sinh^2 \rho = 0 , \quad (3.3)$$

$$2 \sinh \rho \dot{\rho} \dot{t} + \cosh \rho \ddot{t} = 0 , \quad \ddot{\rho} + \sinh \rho \cosh \rho (m^2 + \dot{t}^2) = 0 . \quad (3.4)$$

The first equation in (3.4) can be integrated and expressed in terms of the integral of motion  $\mathcal{E}_0$  (global AdS energy)

$$\dot{t} = \frac{\mathcal{E}_0}{\cosh^2 \rho} . \quad (3.5)$$

Then the conformal gauge constraint becomes

$$\dot{\rho}^2 - \frac{\mathcal{E}_0^2}{\cosh^2 \rho} + m^2 \sinh^2 \rho = 0 , \quad (3.6)$$

---

<sup>6</sup>Notice also that after a shift of  $x$  by  $i\mathbb{K}'$  the potential is real and singular. This is another way to show that its spectrum is real.

with its derivative implying the second-order equation for  $\rho$ . Its solution with  $\rho(0) = 0$  is<sup>7</sup>

$$\sinh \rho(\tau) = \sqrt{\frac{-R_+ R_-}{R_+ - R_-}} \operatorname{sd} \left( m \sqrt{R_+ - R_-} \tau \mid \frac{R_+}{R_+ - R_-} \right), \quad (3.7)$$

$$R_{\pm} = \frac{-m \pm \sqrt{m^2 + 4\mathcal{E}_0^2}}{2m}. \quad (3.8)$$

An alternative form of the solution is

$$\sinh \rho(\tau) = \sqrt{R_+} \operatorname{cn} (x + \mathbb{K}(k^2) \mid k^2), \quad (3.9)$$

$$x = m \sqrt{R_+ - R_-} \tau \equiv w \tau, \quad (3.10)$$

$$k^2 = \frac{R_+}{R_+ - R_-} = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1 + \left( \frac{2\mathcal{E}_0}{m} \right)^2}} \right). \quad (3.11)$$

The induced metric and its curvature are found to be

$$ds^2 = m^2 \sinh^2 \rho \, \eta_{ab}, \quad R^{(2)} = -2 - \frac{2\mathcal{E}_0^2}{m^2 \sinh^4 \rho}. \quad (3.12)$$

The oscillation number is defined as follows

$$N = \sqrt{\lambda} \mathcal{N}, \quad \mathcal{N} = \frac{1}{2\pi} \oint d\rho \, \dot{\rho} = \frac{2\sqrt{\lambda}}{\pi} \int_0^{\rho_{max}} d\rho \sqrt{\frac{\mathcal{E}_0^2}{\cosh^2 \rho} - m^2 \sinh^2 \rho}. \quad (3.13)$$

Changing the variable to  $x = \sinh \rho$  we get

$$\begin{aligned} \mathcal{N} &= \frac{2m\sqrt{\lambda}}{\pi} \int_0^{\sqrt{R_+}} \frac{dx}{1+x^2} \sqrt{(R_+ - x^2)(x^2 - R_-)} \\ &= \frac{2m\sqrt{\lambda}}{\pi} \frac{1}{\sqrt{-R_-}} \left[ R_- \mathbb{E}(q) + (1 + R_+) [\mathbb{K}(q) - (1 + R_-) \Pi(-R_+, q)] \right], \quad q = \frac{R_+}{R_-}. \end{aligned} \quad (3.14)$$

In the short string limit when  $\mathcal{E}_0$  and  $\mathcal{N}$  are small we find (cf. (2.9))<sup>8</sup>

$$\mathcal{N} = \frac{\mathcal{E}_0^2}{2m} - \frac{5\mathcal{E}_0^4}{16m^3} + \frac{63\mathcal{E}_0^6}{128m^5} - \frac{2145\mathcal{E}_0^8}{2048m^7} + \dots, \quad (3.15)$$

$$\mathcal{E}_0 = \sqrt{2m\mathcal{N}} \left( 1 + \frac{5\mathcal{N}}{8m} - \frac{77\mathcal{N}^2}{128m^2} + \frac{1365\mathcal{N}^3}{1024m^3} + \dots \right). \quad (3.16)$$

<sup>7</sup>The equation for  $x = \sinh \rho$  is  $\dot{x}^2 = m^2(x^2 - R_-)(R_+ - x^2)$  and can be compared with the differential equation for the elliptic function  $\operatorname{sd}(z|m)$ .

<sup>8</sup>For large  $\mathcal{N}$  we get Eq. (3.19) of [10] (see also Appendix C of [19]). At small  $\mathcal{N}$  one should get (C.14) of [19],  $E = \sqrt{4\sqrt{\lambda}\mathcal{N}} + \dots$ . These relations show that the definition of  $\mathcal{N}$  in [19] as well as in [10] is off by a factor of 2. Indeed, in the flat space case one should have  $E = \sqrt{2\sqrt{\lambda}\mathcal{N}} + \dots$ , where  $T = \frac{\sqrt{\lambda}}{2\pi}$  is the tension. This  $N$  is identified with the total oscillator number which in the closed string case has to be even. For instance, in the bosonic string case  $\alpha' p^2 = M^2 = 2(N_L + N_R - 2) = 2(N - 1)$  with  $N = N_L + N_R$ .

### 3.2 Quadratic fluctuation Lagrangian

Using the conformal gauge we get 5 massless modes in  $S^5$  and the following bosonic quadratic fluctuation Lagrangian for  $AdS_5$  modes

$$\begin{aligned} \tilde{L} = & \frac{1}{2} \left[ \sinh^2 \rho [(\partial_a \tilde{\beta}_1)^2 - m^2 \tilde{\beta}_1^2] + \sinh^2 \rho \cos^2 m\sigma (\partial_a \tilde{\beta}_3)^2 \right. \\ & + \sinh^2 \rho (\partial_a \tilde{\beta}_2)^2 - \cosh^2 \rho (\partial_a \tilde{t})^2 + 4\mathcal{E}_0 \tanh \rho \tilde{\rho} \partial_\tau \tilde{t} \\ & \left. + (\partial_a \tilde{\rho})^2 + (m^2 + \frac{\mathcal{E}_0^2}{\cosh^4 \rho}) \cosh(2\rho) \tilde{\rho}^2 + 2m \sinh(2\rho) \tilde{\rho} \partial_\sigma \tilde{\beta}_2 \right] \end{aligned} \quad (3.17)$$

After the field redefinitions (here  $\rho = \rho(\tau)$ , cf. (3.2))

$$\cos m\sigma \sinh \rho \tilde{\beta}_3 = \eta_1, \quad \sinh \rho \tilde{\beta}_1 = \eta_2, \quad \cosh \rho \tilde{t} = \zeta, \quad \sinh \rho \tilde{\beta}_2 = \chi, \quad (3.18)$$

the fluctuation Lagrangian becomes ( $i = 1, 2$ )

$$\begin{aligned} \tilde{L} = & \frac{1}{2} \left[ (\partial_a \eta_i)^2 + 2m^2 \sinh^2 \rho \eta_i^2 + (\partial_a \chi)^2 + m^2 (2 \sinh^2 \rho + 1) \chi^2 + 4m \cosh \rho \tilde{\rho} \partial_\sigma \chi \right. \\ & + (\partial_a \tilde{\rho})^2 + (m^2 + \frac{\mathcal{E}_0^2}{\cosh^4 \rho}) \cosh(2\rho) \tilde{\rho}^2 - (\partial_a \zeta)^2 + (\frac{\mathcal{E}_0^2}{\cosh^4 \rho} - 2m^2 \sinh^2 \rho) \zeta^2 \\ & \left. - 4\mathcal{E}_0 \frac{\sinh \rho}{\cosh^2 \rho} \zeta \partial_\tau \tilde{\rho} - 4\mathcal{E}_0 \frac{\dot{\rho}}{\cosh^3 \rho} \zeta \tilde{\rho} \right]. \end{aligned} \quad (3.19)$$

Like in the folded string case in  $AdS_3$  the fluctuation  $\tilde{\rho}$  couples to two other fluctuations. As in the pulsating or folded string cases, to decouple the fluctuations one needs to use the Virasoro constraints expanded at first order in the fluctuations or to use the static gauge on the fluctuations. In the latter case we get 5 massless and 2+1 massive modes with the following Lagrangian (see Appendix C for details)

$$\tilde{L} = \frac{1}{2} \left[ (\partial_a \eta_i)^2 + 2m^2 \sinh^2 \rho \eta_i^2 + (\partial_a \psi)^2 + \left( 2m^2 \sinh^2 \rho - \frac{2\mathcal{E}_0^2}{\sinh^2 \rho} \right) \psi^2 \right]. \quad (3.20)$$

The fermionic fluctuation operator that follows from (2.14) is given by (cf. (2.15); see Appendix C)

$$D'_F = \Gamma_0 \partial_\tau - \Gamma_3 \partial_\sigma + m \Gamma_{124} \sinh \rho. \quad (3.21)$$

Squaring it and diagonalizing we get (cf. (2.17))

$$(D'_F)^2 \rightarrow \mathcal{O}_F = -\partial_\tau^2 + \partial_\sigma^2 - m^2 \sinh^2 \rho \pm im \cosh \rho \dot{\rho}. \quad (3.22)$$

As for the  $\mathbb{R} \times S^2$  pulsating string we can then check UV finiteness either by computing the sum of the squares of effective masses in conformal gauge (absorbing the mixing terms in the covariant derivatives), or by computing the sum of the squares of masses in the static gauge, checking that the result is proportional to the Euler number density. In the conformal gauge we find that the non-zero

(mass)<sup>2</sup> terms are

$$\begin{aligned}
\eta & : 2m^2 \sinh^2 \rho \\
\xi & : 2m^2 \sinh^2 \rho \\
\chi & : m^2(2 \sinh^2 \rho + 1) - \frac{1}{16}(4m \cosh \rho)^2 \\
\rho & : (m^2 + \frac{\mathcal{E}_0^2}{\cosh^2 \rho}) \cosh(2\rho) - \frac{1}{16}(4\mathcal{E}_0 \frac{\sinh \rho}{\cosh^2 \rho})^2 - \frac{1}{16}(4m \cosh \rho)^2 \\
\zeta & : -(\frac{\mathcal{E}_0^2}{\cosh^4 \rho} - 2m^2 \sinh^2 \rho) - \frac{1}{16}(4\mathcal{E}_0 \frac{\sinh \rho}{\cosh^2 \rho})^2 \\
F & : -8m^2 \sinh^2 \rho
\end{aligned} \tag{3.23}$$

which indeed sum up to zero.<sup>9</sup> In the static gauge the non-zero (mass)<sup>2</sup> terms are

$$\begin{aligned}
2 & \times 2m^2 \sinh^2 \rho \\
1 & \times 2m^2 \sinh^2 \rho - \frac{2\mathcal{E}_0^2}{\sinh^2 \rho} \\
-8 & \times m^2 \sinh^2 \rho
\end{aligned} \tag{3.24}$$

and their sum has the same value as in (2.20)

$$-2(m^2 \sinh^2 \rho + \frac{\mathcal{E}_0^2}{\sinh^2 \rho}) = \sqrt{-g} R^{(2)} . \tag{3.25}$$

Again, the  $\tau$ -integral of this term vanishes upon taking into account the boundary contributions (the world-sheet topology is that of a cylinder that has zero Euler number).

### 3.3 Lamé form of fluctuation operators

Let us now show that as in the previous section the fluctuation operators in the static gauge (3.20),(3.22) can be put into the standard Lamé form.

The bosonic operator with mass  $M^2 = 2m^2 \sinh^2 \rho$  in (3.20) can be put into the type I Lamé form as follows ( $\partial_\sigma \rightarrow i\partial_\tau$ )

$$\mathcal{O}_I = w^2 [-\partial_x^2 + 2k^2 \text{sn}^2(x|k^2) - \Lambda] , \tag{3.26}$$

$$x \equiv w\tau + \mathbb{K}(k^2) , \quad \Lambda = 2k^2 + \frac{n^2}{w^2} , \tag{3.27}$$

$$k^2 = \frac{R_+}{R_+ - R_-} = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1 + (\frac{2\mathcal{E}_0}{m})^2}} \right) . \tag{3.28}$$

The operator corresponding to the bosonic fluctuation in (3.20) with the mass  $M^2 = 2m^2 \sinh^2 \rho$  –

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<sup>9</sup>We used that to compute the trace of the square of the mass matrix we may ignore the contributions from mixing terms without derivatives of the fluctuating fields. Also, we took into account that the time-like fluctuation  $\zeta$  has the opposite (ghost) sign of the kinetic term so that  $\zeta \rightarrow i\zeta$  is required in order to bring it to canonical normalization, i.e. we should set  $m_\zeta^2 \zeta^2 \rightarrow -m_\zeta^2 \zeta^2$ .

$\frac{2\mathcal{E}_0^2}{\sinh^2 \rho}$  can be written as

$$\mathcal{O}_{II} = w^2 \left[ -\partial_x^2 + 2k^2 \text{sn}^2(x + \mathbb{K}|k^2) + 2k^2 \text{sn}^2(x + i\mathbb{K}'|k^2) - A \right], \quad (3.29)$$

$$x \equiv w \tau, \quad A \equiv 4k^2 + \frac{n^2}{m^2} \frac{k^2}{R_+} = 4k^2 + \frac{n^2}{w^2}, \quad (3.30)$$

$$k^2 = \frac{R_+}{R_+ - R_-} = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1 + \left( \frac{2\mathcal{E}_0}{m} \right)^2}} \right). \quad (3.31)$$

If written in terms of the Weierstrass function the operator  $\mathcal{O}_{II}$  is of the finite gap Lamé form [21]. Explicitly, by using again a combination of a Landen transformation and a Jacobi imaginary transformation it can be put into the standard single-gap Lamé form (2.21)

$$\mathcal{O}_{II} = \frac{w^2}{\sqrt{1-p^2}} \left[ -\partial_{\tilde{x}}^2 + 2p^2 \text{sn}^2(\tilde{x}|p^2) + B \right], \quad (3.32)$$

$$\tilde{x} \equiv \frac{w \tau}{(1-p^2)^{1/4}} + i \mathbb{K}'(p^2), \quad B = (2-A) \sqrt{1-p^2} - 2, \quad (3.33)$$

$$p^2 = 4 \left[ -2k^2(k^2-1) - k(2k^2-1) \sqrt{k^2-1} \right]. \quad (3.34)$$

Finally, the fermionic fluctuation operator in (3.22) can be written as

$$\mathcal{O}_{III} = w^2 \left[ -\partial_x^2 - k^2 \text{cn}^2(x|k^2) \mp i k \text{sn}(x|k^2) \text{dn}(x|k^2) - \frac{n^2}{w^2} \right], \quad (3.35)$$

$$x = w \tau + \mathbb{K}(k^2), \quad k^2 = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{1 + \left( \frac{2\mathcal{E}_0}{m} \right)^2}} \right). \quad (3.36)$$

Note that this operator is precisely of the same form as the fermionic operator (2.36) in the  $\mathbb{R} \times \mathbb{S}^2$  case, with  $m$  replaced by  $w$ . Thus it can also be transformed into a single-gap Lamé operator.

## 4 Semiclassical quantization of time-periodic solutions of integrable systems

As a preparation for computation of 1-loop correction to the energy of the pulsating strings here we shall briefly review the semiclassical quantization of general (classically integrable) Hamiltonian systems. We shall focus, in particular, on the time-periodic case (see [8] for details and references).

### 4.1 Bohr-Sommerfeld-Maslov quantization

Let us consider a classical Hamiltonian system on a space  $X$  ( $\dim X = n$ ) with a Hamiltonian  $H : T^*X \rightarrow \mathbb{R}$ . We shall assume that its quantum version is a self-adjoint operator  $\hat{H}$  on such that for  $\hbar \rightarrow 0$  it reduces to  $H$ .

The classical integrability requires the existence of  $n$  functions  $F_1, \dots, F_n \in C(T^*X)$  such that: (i)  $dF_1 \wedge \dots \wedge dF_n \neq 0$ , almost everywhere, (ii)  $\{F_i, F_j\} = 0$ , and (iii)  $H = H(F_1, \dots, F_n)$ . This implies that the level sets define  $n$ -tori (Liouville tori) foliating  $T^*X$  and invariant under the Hamiltonian flow. This allows one to define the action variables  $I_i$  parametrizing the foil base and the angle variables  $\varphi_i$ , the coordinates of the torus.

*Semiclassical* integrability requires the existence of quantum extensions  $\widehat{F}_i$  of  $F_i$  ( $\widehat{F}_i \xrightarrow{\hbar \rightarrow 0} F_i$ ) such that in addition to the condition (i) above they satisfy (ii')  $[\widehat{F}_i, \widehat{F}_j] = \mathcal{O}(\hbar^3)$  and (iii')  $\widehat{H} = H(\widehat{F}_1, \dots, \widehat{F}_n) + \mathcal{O}(\hbar^2)$ . Note that  $\widehat{H}$  is well defined without ordering problems because of the condition (ii'). In general, the condition of *quantum* integrability is stronger as it requires  $[\widehat{F}_i, \widehat{F}_j] = 0$ .

Suppose we want to solve the joint diagonalization problem

$$\widehat{F}_i \psi = f_i \psi + \mathcal{O}(\hbar^2) . \quad (4.1)$$

Under some technical simplifying assumption, a WKB-like solution exists if and only if the following Bohr-Sommerfeld-Maslov (BSM) quantization condition is satisfied [22]

$$\frac{1}{2\pi\hbar} \int_{\gamma_i} p \cdot dq = N_i + \frac{\mu_i}{4} + \mathcal{O}(\hbar), \quad i = 1, \dots, n , \quad (4.2)$$

where the integers  $N_i$  thus define the action variables. Here  $\{\gamma_i\}$  is a basis of cycles of a Liouville torus and the Maslov indices  $\mu_i$  take into account the critical points of the cycles. They generalize the familiar 1/2 shift in the standard WKB quantum mechanics relation where  $\mu = 2$  is the number of inversion points.

If the classical invariant torus has only  $p < n$  non trivial cycles, then it can be shown that a simple change in the BSM quantization condition is required. It takes into account the fluctuations transverse to the codimension  $p$  invariant torus. In this case the quantization condition becomes

$$\frac{1}{2\pi\hbar} \int_{\gamma_k} p \cdot dq = N_k + \frac{\mu_k}{4} + \sum_{\alpha=p+1}^n \left(n_\alpha + \frac{1}{2}\right) \frac{\nu_\alpha^{(k)}}{2\pi} + \mathcal{O}(\hbar), \quad \begin{matrix} k = 1, \dots, p, \\ n_\alpha \ll N_k \end{matrix} \quad (4.3)$$

The *stability angles*<sup>10</sup>  $\nu_\alpha^{(k)}$  are found by studying the stability of small fluctuations around the invariant torus (the condition  $n_\alpha \ll N_k$  is necessary in order to be able to use the linearised analysis).

These general considerations can be applied to semiclassical quantization of finite  $g$ -gap solutions of string theory [8]. One starts with a classical energy as a function of the action variables and then simply shifts them according to the BSM quantization conditions, i.e.<sup>11</sup>

$$\begin{aligned} E = E_{cl} \Big( N_1 \hbar + \frac{\hbar}{2} + \hbar \sum_{\alpha=g+2}^{\infty} \left(n_\alpha + \frac{1}{2}\right) \frac{\nu_\alpha^{(1)}}{2\pi}, \dots, \\ N_{g+1} \hbar + \frac{\hbar}{2} + \hbar \sum_{\alpha=g+2}^{\infty} \left(n_\alpha + \frac{1}{2}\right) \frac{\nu_\alpha^{(g+1)}}{2\pi} \Big) + \mathcal{O}(\hbar^2) . \end{aligned} \quad (4.4)$$

In particular, for the ground state ( $n_\alpha = 0$ ) of a 1-gap superstring time-dependent solution of period  $\mathcal{T}$ , we can write (here  $\hbar = \frac{1}{\sqrt{\lambda}}$ ,  $\mathcal{N} = \frac{N}{\sqrt{\lambda}}$ ,  $\mathcal{E} = \frac{E}{\sqrt{\lambda}}$ )

$$\mathcal{E} = \mathcal{E}_{cl}(\mathcal{N}) + \frac{1}{2\sqrt{\lambda}} \frac{1}{\mathcal{T}} \sum_{\nu_s > 0} \nu_s + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) . \quad (4.5)$$

Here  $\mathcal{T}$  is the period of the solution which is the inverse of  $\frac{dE}{dN}$ .<sup>12</sup>

<sup>10</sup>See below for their precise definition.

<sup>11</sup>For finite gap solutions the Maslov indices are all equal to 2.

<sup>12</sup>We took into account that for the superstring the balance of the number of the bosonic and fermionic fluctuations implies the cancellation of the 1/2 shifts. We considered a single  $N$  in (4.4) and expanded in small stability angles. Additional details can be found after Theorem 10.3.1 in the second reference of [8].

In an integrable system, the stability angles may be computed directly since we can solve exactly the problem of evolution of a small perturbation. This is due to the existence of a non-linear superposition principle that allows one to add a “small” solution on top of a soliton background by Backlund transformations. The same construction can be carried over by adding a small additional cut to a finite cut solution of the corresponding integral equations implied by the Bethe equations (or more generally, by considering a genus  $g + 1$  algebraic curve infinitesimally near its genus  $g$  degeneration point).

The details of such constructions (see [23]) may be quite involved and it is of interest to see how they compare with the more standard approach based on second-order differential operators of small fluctuations near a solitonic solution. The classical integrability of the original system should translate into the special properties of the corresponding fluctuation operators given that they appear upon linearization of the same classical equations (see also [6]).

## 4.2 Relation to the Dashen-Hasslacher-Neveu quantization prescription

Let us now demonstrate the relation between the above approach and the Dashen-Hasslacher-Neveu (DHN) approach [7, 24].<sup>13</sup> DHN claimed that one is to impose the condition (cf. (4.3))

$$\oint p \cdot dq + \sum_s (n_s + \frac{1}{2}) (\mathcal{T} \frac{d\nu_s}{d\mathcal{T}} - \nu_s) = 2\pi N . \quad (4.6)$$

Here we set  $\hbar = 1$  and denoted the stability angles by  $\nu_s$  which depend on the period  $\mathcal{T}$ . In this condition the second term in the l.h.s. has to be considered as a perturbation (it represents the one-loop correction in our case). In the classical approximation the first term is a function of the energy  $E$

$$\oint p \cdot dq = I(E) . \quad (4.7)$$

Then the inversion of the relation  $I(E) = 2\pi N$  determines the classical dependence of the energy on the action variable  $N$

$$E_{cl}(N) = I^{-1}(2\pi N) . \quad (4.8)$$

For example, in the case of the pulsating string in  $\mathbb{R} \times S^2$ , discussed in Section 2, the classical period is  $\mathcal{T} = \frac{4}{m} \mathbb{K}\left(\frac{\kappa^2}{m^2}\right)$ , and the classical action evaluated on the classical solution is

$$S(\mathcal{T}) = \sqrt{\lambda} \int_0^{\mathcal{T}} d\tau L = 2m \sqrt{\lambda} \left[ 2\mathbb{E}\left(\frac{\kappa^2}{m^2}\right) + \left(\frac{\kappa^2}{m^2} - 2\right) \mathbb{K}\left(\frac{\kappa^2}{m^2}\right) \right] . \quad (4.9)$$

Now let us make the Legendre transform from  $S$  to  $I \equiv \oint p \cdot dq = S - \mathcal{T} \frac{dS}{d\mathcal{T}}$ . Using (4.9), a short computation leads to the simple result:  $dS/d\mathcal{T} = (dS/d\kappa^2)/(d\mathcal{T}/d\kappa^2) = -\sqrt{\lambda} \kappa^2/2$ . Therefore, we find

$$I = 4m \sqrt{\lambda} \left[ \mathbb{E}\left(\frac{\kappa^2}{m^2}\right) + \left(\frac{\kappa^2}{m^2} - 1\right) \mathbb{K}\left(\frac{\kappa^2}{m^2}\right) \right] = 2\pi N , \quad (4.10)$$

where  $N$  is the adiabatic invariant defined in (2.6).

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<sup>13</sup>See also [25] for applications in the present string-theory context.

According to DHN [7], the quantum correction to the classical relation (4.8) can be expressed as follows. For any  $N$  in (4.6) the associated energy  $E_q$  of a quantum state is

$$E_q = E(N) + \sum_s (n_s + \frac{1}{2}) \frac{d\nu_s}{d\mathcal{T}} . \quad (4.11)$$

where  $E(N)$  is to be obtained from (4.6) by inverting  $N = N(E)$ .<sup>14</sup>

We can then solve (4.6) perturbatively, i.e.  $E(N) = E_{cl}(N) + \delta E$ , so that

$$\left. \frac{dI}{dE} \right|_{E_{cl}} \delta E + \sum_s (n_s + \frac{1}{2}) (\mathcal{T} \frac{d\nu_s}{d\mathcal{T}} - \nu_s) = 0. \quad (4.12)$$

Using this in (4.11) we get, expanding to first order,

$$\begin{aligned} E_q &= E_{cl}(N) - \frac{1}{\mathcal{T}} \sum_s (n_s + \frac{1}{2}) (\mathcal{T} \frac{d\nu_s}{d\mathcal{T}} - \nu_s) + \sum_s (n_s + \frac{1}{2}) \frac{d\nu_s}{d\mathcal{T}} \\ &= E_{cl}(N) + \frac{1}{\mathcal{T}} \sum_s (n_s + \frac{1}{2}) \nu_s . \end{aligned} \quad (4.13)$$

For  $n_s = 0$  (i.e. for the “ground state”) this is the same as the above expression (4.5) found in [8] and references therein. It has an advantage of not involving derivatives of the stability angles over the period. This is the expression we shall use below. An alternative heuristic derivation of (4.5) using quantum field theory methods is sketched in Appendix E.

## 5 One-loop correction to energy of pulsating string solutions

As discussed in the previous section, to compute the one-loop correction to the energy of pulsating string solutions we need to find the stability angles for the Lamé-type fluctuation operators given in sections 2 and 3.

In general, given the 1-d spectral problem with a periodic potential

$$[-\partial_x^2 + V(x)] f(x) = \Lambda f(x) , \quad V(x + \mathcal{T}) = V(x) , \quad (5.1)$$

its two independent solutions  $f_{\pm}(x) = e^{\pm i p(\Lambda) x} \chi_{\pm}(x)$ ,  $\chi_{\pm}(x + \mathcal{T}) = \chi_{\pm}(x)$  satisfy

$$f_{\pm}(x + \mathcal{T}) = e^{\pm i \nu} f_{\pm}(x) , \quad \nu = p\mathcal{T} , \quad (5.2)$$

where  $\nu$  is the “stability angle” and  $p$  is the “quasi-momentum” (in general,  $p$  is a function of  $\Lambda, \mathcal{T}$  and a functional of  $V$ ).

As noted in Sections 2 and 3, the relevant fluctuation operators studied in this paper are all of the single-gap Lamé form. Thus, from the quasi-periodicity properties of the explicit elliptic function solutions discussed in Section 2.3, we find exact expressions for the associated stability angles. These relations involve an auxiliary spectral parameter  $\alpha$ , and solving for this parameter in an explicit way appears to be complicated in general. Here we will concentrate on the “short string” or “near-flat-space” (small oscillation/energy) expansions of the exact relations.

<sup>14</sup>The l.h.s. in (4.6) depends on the parameter  $E$ . The difference between  $E(N)$  and  $E_{cl}$  is due to the second term in the l.h.s. in (4.6).



## 5.1 Pulsating string in $\mathbb{R} \times S^2$

Let us recall that the period of the problem in section 2.1 is  $\mathcal{T} = \frac{4\mathbb{K}}{m}$ . The short string limit is the small  $\kappa$  limit, in which the semiclassical oscillation parameter  $\mathcal{N}$  in (2.7) is small. Below we shall consider the positive of the two possible stability angles differing by sign (see [24]). We shall also rescale the stability angles by a factor of  $2\pi$ .

Once stability angles are computed, we will combine them according to (4.5) to find the correction to the 2d energy. Since in the present case the  $AdS$  time  $t$  and 2d time  $\tau$  are related as in (2.1), i.e.  $t = \kappa\tau$ , there will be similar proportionality of the periods, and the space-time energy and the 2d energy will be related by

$$E_{\text{spacetime}} = \frac{1}{\kappa} E_{2d} . \quad (5.3)$$

### 5.1.1 Stability angles

The 4 massless  $AdS_5$  fluctuations in (2.10) have the obvious stability angle

$$\nu_{AdS_5} = 4\mathbb{K} \sqrt{k^2 + \frac{n^2}{m^2}} , \quad k \equiv \frac{\kappa}{m} . \quad (5.4)$$

Expanding in small  $\kappa$ , i.e. in small  $k$ , we get

$$\begin{aligned} \nu_{AdS_5} = & \frac{2\pi n}{m} + k^2 \left( \frac{\pi m}{n} + \frac{\pi n}{2m} \right) + \frac{\pi k^4 (-8m^4 + 8m^2 n^2 + 9n^4)}{32mn^3} \\ & + \frac{\pi k^6 (16m^6 - 8m^4 n^2 + 18m^2 n^4 + 25n^6)}{128mn^5} + \dots . \end{aligned} \quad (5.5)$$

The  $S^5$  bosonic fluctuations (both Type I (2.26) and Type II (2.31)) are associated with the standard Lamé equation and the stability angle is

$$\nu_{S^5} = \pm 4\mathbb{K} \left( i\mathbb{Z}(\alpha | k^2) + \frac{\pi}{2\mathbb{K}} \right) \equiv \pm 4\mathbb{K} i\mathbb{Z}(\alpha | k^2) , \quad (5.6)$$

$$\text{sn}(\alpha | k^2) = \sqrt{\frac{1 + k^2 - \Lambda}{k^2}} = \frac{1}{k} \sqrt{1 - \frac{n^2}{m^2}} . \quad (5.7)$$

We shall fix the sign in (5.6) by the condition  $\nu > 0$ .

Let us define  $a = \sqrt{1 - \frac{n^2}{m^2}}$  and begin with the case  $|n| < |m|$  which is  $a \in (0, 1)$ . In general (here  $\mathbb{E} = \mathbb{E}(k^2)$ , etc.),

$$\mathbb{Z}(\text{sn}^{-1}(\frac{a}{k} | k^2) | k^2) = \int_0^{a/k} dt \left( \sqrt{\frac{1 - k^2 t^2}{1 - t^2}} - \frac{\mathbb{E}}{\mathbb{K}} \frac{1}{\sqrt{1 - k^2 t^2} \sqrt{1 - t^2}} \right) . \quad (5.8)$$

In the short string limit  $k \rightarrow 0^+$  limit, and for  $a \in (0, 1)$  we can exploit  $\mathbb{Z}(\text{sn}^{-1}(1 | k^2) | k^2) = 0$ . Taking into account Mathematica conventions for the cuts, we find

$$\mathbb{Z}(\text{sn}^{-1}(\frac{a}{k} | k^2) | k^2) = i \int_1^{a/k} \frac{dt}{\sqrt{t^2 - 1}} \left( \sqrt{1 - k^2 t^2} - \frac{\mathbb{E}}{\mathbb{K}} \frac{1}{\sqrt{1 - k^2 t^2}} \right) . \quad (5.9)$$

The two basic integrals are

$$\int_1^{a/k} \frac{dt}{\sqrt{t^2-1}} \sqrt{1-k^2t^2} = i \left( \mathbb{E} \left( \arcsin \frac{a}{k} \mid k^2 \right) - \mathbb{E} \right), \quad (5.10)$$

$$\int_1^{a/k} \frac{dt}{\sqrt{t^2-1}} \frac{1}{\sqrt{1-k^2t^2}} = i \left( \mathbb{F} \left( \arcsin \frac{a}{k} \mid k^2 \right) - \mathbb{K} \right). \quad (5.11)$$

In order to expand at small  $k$ , we use the transformation

$$\mathbb{E} \left( \arcsin \frac{a}{k} \mid k^2 \right) = \frac{\mathbb{E}}{\mathbb{K}} \mathbb{F} \left( \arcsin \frac{a}{k} \mid k^2 \right) + i \sqrt{1-a^2} \sqrt{1-\frac{k^2}{a^2}} \left( \frac{\Pi(a^2 \mid k^2)}{\mathbb{K}} - 1 \right). \quad (5.12)$$

The final result is remarkably simple since all incomplete elliptic integrals simplify. It reads

$$\mathbb{Z}(\operatorname{sn}^{-1}(\frac{a}{k} \mid k^2) \mid k^2) = i \sqrt{1-a^2} \sqrt{1-\frac{k^2}{a^2}} \left( 1 - \frac{\Pi(a^2 \mid k^2)}{\mathbb{K}} \right). \quad (5.13)$$

This expression can be expanded at small  $k$ . The product with  $\mathbb{K}$  turns out to be

$$\begin{aligned} i \mathbb{K} \mathbb{Z}(\operatorname{sn}^{-1}(\frac{a}{k} \mid k^2) \mid k^2) &= -\frac{1}{2} \pi (\sqrt{1-a^2} - 1) - \frac{\pi}{8} \sqrt{1-a^2} k^2 \\ &\quad - \frac{\pi \sqrt{1-a^2} (9a^2 + 4) k^4}{128a^2} - \frac{\pi \sqrt{1-a^2} (25a^4 + 12a^2 + 8) k^6}{512a^4} + \mathcal{O}(k^8). \end{aligned} \quad (5.14)$$

Using that  $a = \sqrt{1 - \frac{n^2}{m^2}}$  we find (for  $n > 0$ ) after fixing the sign in (5.6) and subtracting constant  $2\pi$  term

$$\begin{aligned} \nu_{s^5} = -4i \mathbb{K} \mathbb{Z}(\operatorname{sn}^{-1}(\frac{a}{k} \mid k^2) \mid k^2) &= \frac{2\pi n}{m} + \frac{\pi k^2 n}{2m} + \frac{\pi k^4 n (13m^2 - 9n^2)}{32m(m-n)(m+n)} \\ &\quad + \frac{\pi k^6 n (45m^4 - 62m^2 n^2 + 25n^4)}{128m(m-n)^2(m+n)^2} + \dots \end{aligned} \quad (5.15)$$

Some comments are in order: (i) the singularity at  $n = m$  is only an apparent one since it happens at  $a = 0$  where our derivation cannot be applied (the above expression following from (5.14) is just zero at that point; this is not a problem since our  $n$  is discrete); (ii) one can compare the l.h.s. of (5.15) with the result (5.11) for imaginary  $a$ , i.e. for  $n > m$ , and the equation still applies.

Let us mention a different method to perform the short-string expansion leading to (5.15): instead of expanding the exact stability angles we shall use direct perturbation theory in small  $\kappa$ . We shall use this perturbative approach below in the case of pulsating string in  $AdS_3$ . Let us see how this works for the Type I operator from (2.27) (here we take  $m = 1$  for simplicity), i.e.

$$\mathcal{O} = -\partial_x^2 + 2\kappa^2 \operatorname{sn}^2(x \mid \kappa^2) - \kappa^2 - n^2. \quad (5.16)$$

Let us introduce the variable  $y$  related to  $x$  by

$$x = \frac{2\mathbb{K}(\kappa^2)}{\pi} y, \quad (5.17)$$

in terms of which the period of the problem is  $\kappa$ -independent and equal to  $2\pi$ . Expanding in  $\kappa \rightarrow 0$  the operator (5.16) can be written as

$$\mathcal{O} = \mathcal{O}_0 + \kappa^2 \mathcal{O}_1 + \kappa^4 \mathcal{O}_2 + \dots, \quad (5.18)$$

$$\mathcal{O}_0 = -\partial_y^2 - n^2, \quad \mathcal{O}_1 = \frac{1}{2}\partial_y^2 - \cos 2y, \quad \mathcal{O}_2 = \frac{3}{32}\partial_y^2 + \sin^2 y \cos^2 y. \quad (5.19)$$

then one easily finds iterative solution of  $\mathcal{O} f = 0$  in series expansion in  $\kappa^2$  starting from the  $f_0 = e^{iny}$  solution of  $\mathcal{O}_0 f_0 = 0$

$$f(y) = e^{iny + \kappa^2 h_1(y) + \kappa^4 h_2(y) + \dots}, \quad (5.20)$$

$$h_1(y) = \frac{in \sin(2y)}{4(n^2 - 1)} - \frac{\cos(2y)}{4(n^2 - 1)} + \frac{iny}{4}, \quad (5.21)$$

$$h_2(y) = \frac{1}{64(n^2 - 1)^2} \left[ 9in^5 y - 22in^3 y + in(n^2 + 1) \sin(4y) + 4i(n^2 - 3)n \sin(2y) - 2n^2 \cos(4y) + 13iny + 8 \cos(2y) \right]. \quad (5.22)$$

The corresponding stability angle is then

$$\nu = -i \log \frac{f(2\pi)}{f(0)} = 2\pi n + \frac{1}{2} \pi \kappa^2 n + \frac{\pi \kappa^4 n(9n^2 - 13)}{32(n^2 - 1)} + \dots, \quad (5.23)$$

which agrees with the expansion of the exact result (5.15) after setting there  $m = 1$  and noting that  $k = \frac{\kappa}{m}$ .

In the case of the fermionic fluctuation operator (2.36) the expression for the stability angle is

$$\nu_F = \pm 4i \mathbb{K} \left[ \frac{1}{2} Z(\alpha(\beta) | k^2) + i \sqrt{\beta} \sqrt{1 + \frac{16\beta k^2}{(1 - 4\beta)^2}} \right], \quad (5.24)$$

where

$$\alpha(\beta) = \text{cn}^{-1} \left( -\frac{1 + 4\beta}{1 - 4\beta} | k^2 \right), \quad \beta = \frac{n^2}{m^2}. \quad (5.25)$$

Since  $\beta$  is independent of  $\kappa$ , we can immediately expand at small  $k$ . Fixing the sign, the result is

$$\begin{aligned} \nu_F = & \frac{2\pi n}{m} + \frac{\pi n(3m^2 + 4n^2)}{2m(2n - m)(m + 2n)} k^2 - \frac{\pi n(15m^6 - 276m^4 n^2 - 304m^2 n^4 + 576n^6)}{32m(m - 2n)^3(m + 2n)^3} k^4 \\ & - \frac{\pi n(35m^{10} - 780m^8 n^2 + 9696m^6 n^4 + 9856m^4 n^6 - 28928m^2 n^8 + 25600n^{10})}{128m(m - 2n)^5(m + 2n)^5} k^6 + \dots \end{aligned} \quad (5.26)$$

It is interesting to understand the singularity of the expansion at  $n = \frac{m}{2}$  or  $\beta = \frac{1}{4}$ . If we plot the fermionic discriminant as  $k$  decreases we can see that there is an antiperiodic solution appearing at  $\beta = \frac{1}{4}$  when  $k = 0$  and being absent for  $k > 0$ . This curious phenomenon is the reason for the singularity of the small  $k$  expansion. One can just regulate this “resonance” by taking  $m$  to be odd.

### 5.1.2 Sum of stability angles and short string expansion of the energy

Let us now combine the above fluctuation frequencies expanded in powers of  $\kappa = km$  (5.5), (5.15) and (5.26) with proper multiplicities and signs as they should appear in the 1-loop correction to the energy in (4.5)

$$\begin{aligned}
\nu_n &= 4 \times (\nu_{AdS_5} + \nu_{S^5}) - 8 \times \nu_F \\
&= \frac{4\pi\kappa^2 m}{n(m^2 - 4n^2)} - \frac{\pi\kappa^4(2m^8 - 28m^6n^2 + 133m^4n^4 - 128m^2n^6 + 48n^8)}{2mn^3(m^2 - 4n^2)^3(m^2 - n^2)} \\
&\quad + \frac{\pi\kappa^6}{16m^3(m^2 - n^2)^2(m^2n - 4n^3)^5} \left( 8m^{16} - 180m^{14}n^2 + 1705m^{12}n^4 - 8772m^{10}n^6 + 25883m^8n^8 \right. \\
&\quad \left. 35456m^6n^{10} + 25824m^4n^{12} - 13824m^2n^{14} + 3840n^{16} \right) + \dots .
\end{aligned} \tag{5.27}$$

As a check, we observe that the sum over  $n$  of this combination is convergent at large  $n$ .

In the rest of this section we shall focus on the case of  $m = 1$ . Dividing by the period  $4\mathbb{K}(\kappa^2)$  we get

$$\begin{aligned}
\frac{\nu_n}{\mathcal{T}} &= \frac{2}{n - 4n^3} \kappa^2 - \frac{16n^8 - 80n^6 + 115n^4 - 26n^2 + 2}{4n^3(n^2 - 1)(4n^2 - 1)^3} \kappa^4 \\
&\quad - \frac{256n^{16} - 896n^{14} + 2560n^{12} - 5864n^{10} + 5295n^8 - 1954n^6 + 402n^4 - 44n^2 + 2}{8n^5(n^2 - 1)^2(4n^2 - 1)^5} \kappa^6 + O(\kappa^7) .
\end{aligned} \tag{5.28}$$

The sum over modes with  $n > 1$  then gives

$$\begin{aligned}
\frac{1}{\mathcal{T}} \sum_{n=2}^{\infty} \nu_n &= \left( \frac{8}{3} - 4 \log 2 \right) \kappa^2 + \left( \frac{3\zeta_3}{8} - \frac{347}{432} + \frac{\log 2}{2} \right) \kappa^4 \\
&\quad + \left( -\frac{63\zeta_3}{64} - \frac{15\zeta_5}{64} + \frac{38759}{31104} + \frac{\log 2}{4} \right) \kappa^6 + O(\kappa^8) .
\end{aligned} \tag{5.29}$$

where we have used the shorthand notation for the Riemann zeta function:  $\zeta_k = \zeta(k)$ . The  $n = 0$  contribution comes only from  $AdS_5$  part and we get

$$\nu_0 = 16 \mathbb{K}(\kappa^2) \kappa , \quad \frac{\nu_0}{\mathcal{T}} = 4\kappa . \tag{5.30}$$

The  $n = 1$  contribution comes only from the  $AdS_5$  part and the fermions

$$\frac{\nu_1}{\mathcal{T}} = -\frac{5\kappa^2}{3} + \frac{401\kappa^4}{432} - \frac{18529\kappa^6}{15552} + O(\kappa^8) . \tag{5.31}$$

Summing up all the contributions we get from (4.5) the following expression for the 1-loop correction to the string energy (taking into account the relation (5.3) valid in the static gauge  $t = \kappa\tau$ )

$$\begin{aligned}
\mathcal{E}_1 = \frac{1}{2\mathcal{T}\kappa} \sum_{n=-\infty}^{\infty} \nu_n &= 2 + \kappa(1 - 4 \log 2) + \frac{1}{8} \kappa^3 \left( 3\zeta_3 + 1 + 4 \log 2 \right) \\
&\quad + \frac{1}{4} \kappa^5 \left( -\frac{63\zeta_3}{16} - \frac{15\zeta_5}{16} + \frac{7}{32} + \log 2 \right) + O(\kappa^7) .
\end{aligned} \tag{5.32}$$

In general, we can organize the short string expansion of the energy as

$$E = E\left(\frac{N}{\sqrt{\lambda}}, \sqrt{\lambda}\right) = \sqrt{\lambda} \mathcal{E}_0(\mathcal{N}) + \mathcal{E}_1(\mathcal{N}) + \frac{1}{\sqrt{\lambda}} \mathcal{E}_2(\mathcal{N}) + \dots, \quad (5.33)$$

$$\mathcal{E}_k = \sqrt{2\mathcal{N}} \left( a_{0k} + a_{1k} \mathcal{N} + a_{2k} \mathcal{N}^2 + \dots \right) + c_{0k} + c_{1k} \mathcal{N} + \dots. \quad (5.34)$$

where  $c_{nk}$  are coefficients of “non-analytic” terms [4]. Using (2.8), (2.9) and (5.32) we thus find that for the pulsating string in  $\mathbb{R} \times S^2$

$$\mathcal{E}_0 = \sqrt{2\mathcal{N}} \left( 1 - \frac{1}{8} \mathcal{N} - \frac{5}{128} \mathcal{N}^2 + \dots \right), \quad (5.35)$$

$$\begin{aligned} E_1 \equiv \mathcal{E}_1 &= 2 + \sqrt{2\mathcal{N}} \left[ 1 - 4 \log 2 + \left( \frac{3}{2} \log 2 + \frac{3}{4} \zeta_3 + \frac{1}{8} \right) \mathcal{N} \right. \\ &\quad \left. + \left( \frac{25}{32} \log 2 - \frac{135}{32} \zeta_3 - \frac{15}{16} \zeta_5 + \frac{11}{128} \right) \mathcal{N}^2 + \dots \right]. \end{aligned} \quad (5.36)$$

The energy can be re-written in terms of  $N$  and the string tension as follows

$$E = \sqrt{2N\sqrt{\lambda}} \left( a_{00} + \frac{a_{10}N + a_{01}}{\sqrt{\lambda}} + \dots \right) + c_{01} + \dots, \quad (5.37)$$

$$a_{00} = 1, \quad a_{10} = -\frac{1}{8}, \quad a_{01} = 1 - 4 \log 2, \quad c_{01} = 2, \quad \dots \quad (5.38)$$

## 5.2 Pulsating string in $AdS_3$

The aim of this subsection is to use the results of section 3 to compute, in a similar way as above, the one-loop correction to the energy in the short string limit of small oscillation parameter  $\mathcal{N} \rightarrow 0$  or small classical energy  $\mathcal{E}_0 \rightarrow 0$ .

In contrast to the pulsating string in  $\mathbb{R} \times S^2$  where we can use the static gauge  $t = \kappa\tau$  in which the relation between the 2d and space-time energy is simple, here this is no longer the case as in the conformal gauge the classical solution for the  $AdS_5$  time  $t$  depends on the world-sheet time  $\tau$  in a non-linear way. Here we may fix the static gauge on the fluctuation of  $t$  (i.e. set it to zero) while using the classical conformal-gauge relation between  $t$  and  $\tau$  in (3.5), i.e.<sup>15</sup>

$$dt = \dot{t} d\tau = \frac{\mathcal{E}_0}{\cosh^2 \rho(\tau)} d\tau. \quad (5.39)$$

Since the relation between  $t$  and  $\tau$  is a change of variable, it does not matter for the equations of motions and the fluctuation operator which can be solved in terms of the  $\tau$  variable. What is affected is the expression for the period, which for the  $t$ -motion is then

$$\mathcal{T} = \mathcal{E}_0 \int_0^{\mathcal{T}_\tau} \frac{d\tau}{\cosh^2 \rho(\tau)}, \quad (5.40)$$

where  $\mathcal{T}_\tau$  is the period in the  $\tau$  variable. Having found the stability angles we should then use again the expression (4.5) where to get the space-time energy we will need to divide by the period  $\mathcal{T}$  in (5.40) corresponding to the variable  $t$ .

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<sup>15</sup>For comparison, the analog of the classical energy parameter  $\mathcal{E}$  in the case of pulsating string on  $S^2$  in the previous subsection where we had  $dt = \kappa d\tau$  was  $\kappa$ .

### 5.2.1 Stability angles

As follows from section 3 the bosonic fluctuations of type I obey the equation

$$\mathcal{O}_I \zeta_n \equiv [-\partial_\tau^2 - n^2 - z \sinh^2 \rho] \zeta_n = 0, \quad (5.41)$$

where  $z = 2$  for the non-trivial boson, and  $z = 0$  for the free modes. The stability angle for  $\mathcal{O}_I$  is

$$\nu_I = \pm 4 \mathbb{K} \left( i \mathbb{Z}(\alpha | k^2) + \frac{\pi}{2\mathbb{K}} \right) \equiv \pm 4 \mathbb{K} i \mathbb{Z}(\alpha | k^2), \quad (5.42)$$

where ( $w$  is defined in (3.10))

$$\text{sn}(\alpha | k^2) = \frac{1}{k} \sqrt{1 - k^2 - \frac{n^2}{w^2}}. \quad (5.43)$$

Expanding in the limit  $\mathcal{E}_0 \rightarrow 0$  (cf. (D.13))

$$\nu = \nu^{(1)} \mathcal{E}_0^2 + \nu^{(2)} \mathcal{E}_0^4 + \nu^{(3)} \mathcal{E}_0^6 + \dots, \quad (5.44)$$

we find for  $|n| \neq 1$  (see Appendix D)

$$\begin{aligned} \nu_{I,n}^{(1)} &= \pi \frac{3n^2 - z}{2n}, \\ \nu_{I,n}^{(2)} &= -\pi \frac{105n^6 - 15n^4(2z + 7) - 3n^2(z - 10)z + 2z^2}{32n^3(n^2 - 1)}, \\ \nu_{I,n}^{(3)} &= \frac{\pi \left( 1155(n^2 - 1)^2 n^6 - (5n^4 - 5n^2 + 2)z^3 - (35n^4 - 65n^2 + 24)n^2 z^2 - 315(n^2 - 1)^2 n^4 z \right)}{128n^5(n^2 - 1)^2}. \end{aligned} \quad (5.45)$$

The singularity at  $n = \pm 1$  is absent for  $z = 0$ . In the  $z = 2$  case, a more detailed analysis shows that at the considered orders we have  $\nu_{I,\pm 1}^{(2)} = 0$ .

In the  $z = 2$  case, there is a singularity at  $n = 0$ . At leading order the problem is related to the fact that the stability angle  $\nu_0$  of the equation

$$(-\partial_y^2 - 2\mathcal{E}_0^2 \sin^2 y) \zeta(y) = 0, \quad \zeta(a + 2\pi) = e^{i\nu_0} \zeta(a), \quad (5.46)$$

goes like  $\mathcal{E}_0$  for  $\mathcal{E}_0 \rightarrow 0$ . This is valid in general and can be checked numerically in the present case. The precise result in our case is

$$\nu_{I,0} = 2\pi\mathcal{E}_0 + \dots \quad (5.47)$$

In the case of type II fluctuation equation we have for  $m = 1$

$$\mathcal{O}_{II} \zeta_n \equiv \left[ -\partial_\tau^2 - n^2 + \frac{2\mathcal{E}_0^2}{\sinh^2 \rho} - 2 \sinh^2 \rho \right] \zeta_n = 0, \quad (5.48)$$

where

$$\sinh \rho(\tau) = \sqrt{\frac{-R_+ R_-}{R_+ - R_-}} \text{sd} \left( \sqrt{R_+ - R_-} \tau \mid \frac{R_+}{R_+ - R_-} \right). \quad (5.49)$$

As we have argued in section 3.3 this operator is of the Lamé type. The stability angle is then

$$\nu_{II} = \pm 4 \mathbb{K}(k^2) \left( i \mathbb{Z}(\alpha | p^2) + \frac{\pi}{2\mathbb{K}(p^2)} \right), \quad (5.50)$$

where ( $B$  and  $p$  were defined in (3.33),(3.34))

$$\text{sn}(\alpha | p^2) = \sqrt{\frac{1 + p^2 - B}{p^2}}. \quad (5.51)$$

The coefficients in the small  $\mathcal{E}$  expansion of the stability angle are explicitly

$$\begin{aligned} \nu_{II,n}^{(1)} &= \pi \frac{n(3n^2 - 7)}{2(n^2 - 1)}, \\ \nu_{II,n}^{(2)} &= -\pi \frac{n(105n^6 - 435n^4 + 603n^2 - 337)}{32(n^2 - 1)^3}, \\ \nu_{II,n}^{(3)} &= -\frac{\pi n(1155n^{10} - 7035n^8 + 17150n^6 - 21430n^4 + 14159n^2 - 4511)}{128(n^2 - 1)^5}. \end{aligned} \quad (5.52)$$

The same result can be obtained without knowing the analytical expression for  $\nu_{II}$  and using perturbation theory alone. The detailed calculation is reported for completeness in Appendix D.

The type III (fermionic) fluctuation equation has the form

$$\mathcal{O}_{III} \zeta_n \equiv \left[ -\partial_\tau^2 - n^2 - \sinh^2 \rho \pm i \frac{d}{d\tau} \sinh \rho \right] \zeta_n = 0. \quad (5.53)$$

The stability angle for the fermionic operator  $\mathcal{O}_{III}$  is

$$\nu_{III} = \pm 4 i \mathbb{K} \left[ \frac{1}{2} Z(\alpha(\beta) | k^2) + i \sqrt{\beta} \sqrt{1 + \frac{16\beta k^2}{(1-4\beta)^2}} \right], \quad (5.54)$$

$$\alpha(\beta) = \text{cn}^{-1} \left( -\frac{1+4\beta}{1-4\beta} | k^2 \right), \quad \beta = \frac{n^2}{w^2}. \quad (5.55)$$

Expanding in small  $\mathcal{E}$  we obtain the stability angle (5.44) with coefficients

$$\begin{aligned} \nu_n^{(1)} &= \pi \frac{n(12n^2 - 7)}{2(4n^2 - 1)}, \\ \nu_n^{(2)} &= -\pi \frac{n(6720n^6 - 6960n^4 + 2412n^2 - 337)}{32(4n^2 - 1)^3}, \\ \nu_n^{(3)} &= \frac{\pi n(1182720n^{10} - 1800960n^8 + 1097600n^6 - 342880n^4 + 56636n^2 - 4511)}{128(4n^2 - 1)^5}. \end{aligned} \quad (5.56)$$

### 5.2.2 Sum of stability angles and short string expansion of the energy

Adding together the contributions of the  $AdS_5$  and  $S^5$  bosonic modes (including the 5 massless modes with  $z = 0$ ) and the fermions in (5.45),(5.52),(5.56) we obtain for the small  $\mathcal{E}_0$  expansion of the sum

of individual stability angles (for  $n \geq 2$ )

$$\begin{aligned} \frac{1}{\pi} \nu_n &= 5 \times \left( \frac{3n^2 - 0}{2n} \mathcal{E}_0^2 + \dots \right) + 1 \times \left( \frac{n(3n^2 - 7)}{2(n-1)(n+1)} \mathcal{E}_0^2 + \dots \right) + 2 \times \left( \frac{3n^2 - 2}{2n} \mathcal{E}_0^2 + \dots \right) \\ &\quad - 8 \times \left( \frac{n(12n^2 - 7)}{2n(4n^2 - 1)} \mathcal{E}_0^2 + \dots \right) \\ &= - \frac{2\pi(2n^2 + 1)}{n(n^2 - 1)(4n^2 - 1)} \mathcal{E}_0^2 + \frac{\pi(240n^{12} - 560n^{10} + 713n^8 - 361n^6 + 83n^4 - 8n^2 + 1)}{2n^3(n^2 - 1)^3(4n^2 - 1)^3} \mathcal{E}_0^4 + \dots \end{aligned} \quad (5.57)$$

The  $n = \pm 1$  contributions come from fermions and free bosonic modes

$$2\nu_1 = \frac{5\pi}{3} \mathcal{E}_0^2 + \frac{505\pi}{432} \mathcal{E}_0^4 - \frac{105515}{31104} \mathcal{E}_0^6 + \dots \quad (5.58)$$

Then the total sum for  $n \neq 0$  is

$$\begin{aligned} \sum_{n \neq 0}^{\infty} \nu_n &= \pi(-10 + 16 \log 2) \mathcal{E}_0^2 + \pi \left( \frac{199}{8} - 30 \log 2 - \frac{3}{2} \zeta_3 \right) \mathcal{E}_0^4 \\ &\quad + \left( -\frac{9395\pi}{128} + \frac{315\pi}{4} \log 2 + \frac{111\pi}{16} \zeta_3 + \frac{15}{16} \zeta_5 \right) \mathcal{E}_0^6 + \dots \end{aligned} \quad (5.59)$$

The overall (negative) sign with which this sum enters the expression for the 1-loop energy can be fixed by looking at the contribution of the free  $S^5$  modes. The first correction to the period in the  $\tau$  variable is negative

$$\mathcal{T}_\tau = \frac{4\mathbb{K}(\frac{R_+}{R_+ - R_-})}{\sqrt{R_+ - R_-}} = 2\pi - \frac{3\pi}{2} \mathcal{E}_0^2 + \frac{105\pi}{32} \mathcal{E}_0^4 + \dots \quad (5.60)$$

Adding the zero mode  $n = 0$  contribution of the non-trivial type I fluctuation (5.41) (multiplied by 2 which is the number of bosons with  $z = 2$ ) we find

$$E_1 = \frac{1}{2\mathcal{T}} \left( 2 \cdot 2\pi \mathcal{E}_0 - \sum_{n \neq 0}^{\infty} \nu_n \right), \quad (5.61)$$

where  $\mathcal{T}$  is the period of the  $t$  variable in (5.40)

$$\mathcal{T} = \mathcal{E}_0 \int_0^{\mathcal{T}_\tau} \frac{d\tau}{\cosh^2 \rho} = 2\pi \mathcal{E}_0 - \frac{5}{2} \pi \mathcal{E}_0^3 + \frac{189\pi}{32} \mathcal{E}_0^5 + \dots \quad (5.62)$$

Using the fact that the classical energy parameter is related to the oscillation number as in (3.16) (here  $m = 1$ )

$$\mathcal{E}_0 = \sqrt{2\mathcal{N}} \left( 1 + \frac{5}{8} \mathcal{N} - \frac{77}{128} \mathcal{N}^2 + \dots \right), \quad (5.63)$$

we finally obtain (cf. (5.36))

$$\begin{aligned} E_1 &= 1 + \sqrt{2\mathcal{N}} \left[ \frac{5}{2} - 4 \log 2 + \left( -\frac{37}{8} + \frac{5}{2} \log 2 + \frac{3}{4} \zeta_3 \right) \mathcal{N} \right. \\ &\quad \left. + \left( \frac{3915}{256} - \frac{231}{32} \log 2 - \frac{117}{32} \zeta_3 - \frac{15}{16} \zeta_5 \right) \mathcal{N}^2 + \dots \right]. \end{aligned} \quad (5.64)$$



## 6 One-loop correction to energy of folded string in $\mathbb{R} \times S^2$

To get a better understanding of the structure of energy of “small” semiclassical strings it is useful to supplement the discussion of the folded spinning string in  $AdS_3$  in [3, 6] and the analysis of the pulsating strings in  $\mathbb{R} \times S^2$  and  $AdS_3$  carried out above with a similar study of the 1-loop corrected energy of spinning folded string in  $\mathbb{R} \times S^2$  part of  $AdS_5 \times S^5$ . This will be the aim of this section.

We shall start with the case of the folded string in  $S^3$  moving along big circle with orbital momentum  $J_1 = \sqrt{\lambda} \mathcal{J}_1$  and spinning around its c.o.m. with momentum  $J_2 = \sqrt{\lambda} \mathcal{J}_2$ . When discussing one-loop corrections we will eventually specify to the case of  $\mathcal{J}_1 = 0$  and expand in  $\mathcal{J}_2 \rightarrow 0$ .

### 6.1 Classical solution

Let us start with a brief review of the folded string with two angular momenta moving in  $S^3 \subset AdS_5 \times S^5$  [32]. The metric of  $\mathbb{R} \times S^3$  is with metric

$$ds^2 = -dt^2 + d\theta^2 + \cos^2 \theta d\varphi_1^2 + \sin^2 \theta d\varphi_2^2. \quad (6.1)$$

and the ansatz one assumes is ( $i = 1, 2$ )

$$t = \kappa \tau, \quad \theta = \theta(\sigma), \quad \varphi_i = w_i \tau. \quad (6.2)$$

In conformal gauge the only non-trivial equation of motion reads (we assume  $w_2 > w_1$ )

$$\theta'' + \frac{1}{2} w_{21}^2 \sin(2\theta) = 0, \quad w_{21}^2 = w_2^2 - w_1^2, \quad (6.3)$$

which has the Virasoro condition as its first integral

$$\theta'^2 + w_1^2 \cos^2 \theta + w_2^2 \sin^2 \theta = \kappa^2. \quad (6.4)$$

The periodic solution with  $\theta(0) = 0$  is (see also [20, 12])

$$\sin \theta = \sqrt{q} \operatorname{sn}(w_{21} \sigma | q), \quad \cos \theta = \operatorname{dn}(w_{21} \sigma | q), \quad (6.5)$$

where

$$q = \sin^2 \theta_0 = \frac{\kappa^2 - w_1^2}{w_2^2 - w_1^2}, \quad w_{21} = \sqrt{w_2^2 - w_1^2} = \frac{2}{\pi} \mathbb{K}(q). \quad (6.6)$$

Let us note that the equation (6.4) can be written in a form which depends only on  $q$

$$\theta'^2 = w_{21}^2 (\sin^2 \theta_0 - \sin^2 \theta) = \left[ \frac{2}{\pi} \mathbb{K}(q) \right]^2 (q - \sin^2 \theta), \quad \theta(0) = 0. \quad (6.7)$$

The expressions for the energy and the two angular momenta can be given, e.g., in terms of the hypergeometric functions

$$\mathcal{E}_0 = \kappa, \quad \mathcal{J}_1 = \frac{w_1}{w_{21}} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, q\right), \quad \mathcal{J}_2 = \frac{w_2}{w_{21}} \frac{q}{2} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2, q\right). \quad (6.8)$$

$\mathcal{J}_i$  satisfy the relationship

$$\frac{\mathcal{J}_1}{w_1} + \frac{\mathcal{J}_2}{w_2} = 1. \quad (6.9)$$

Useful relations which allows to eliminate  $q$  and find  $\mathcal{E}_0 = \mathcal{E}_0(\mathcal{J}_1, \mathcal{J}_2)$  is [20]

$$\left(\frac{\mathcal{E}_0}{\mathbb{K}(q)}\right)^2 - \left(\frac{\mathcal{J}_1}{\mathbb{E}(q)}\right)^2 = \frac{4}{\pi^2} q, \quad \left(\frac{\mathcal{J}_2}{\mathbb{K}(q) - \mathbb{E}(q)}\right)^2 - \left(\frac{\mathcal{J}_1}{\mathbb{E}(q)}\right)^2 = \frac{4}{\pi^2}. \quad (6.10)$$

In the short string limit, i.e. the small  $q$  limit, the solution for  $\theta$  can be expanded as

$$\theta(\sigma) = \sqrt{q} \sin \sigma + \left(\frac{3 \sin \sigma}{16} + \frac{\sin(3\sigma)}{48}\right) q^{3/2} + \left(\frac{23 \sin \sigma}{256} + \frac{\sin(3\sigma)}{64} + \frac{\sin(5\sigma)}{1280}\right) q^{5/2} + O(q^{7/2}). \quad (6.11)$$

To expand the energy in small spins (or  $q \rightarrow 0$ ) one may consider two special scaling limits. The first is when

$$\mathcal{J}_2 \rightarrow 0, \quad r = \frac{\mathcal{J}_1}{\mathcal{J}_2} = \text{fixed}. \quad (6.12)$$

Then

$$\begin{aligned} \mathcal{J}_2 &= \frac{q}{2} + \frac{3q^2}{16} + \frac{8r^2 + 15}{128} q^3 + \frac{208r^2 + 175}{2048} q^4 + O(q^5), \\ w_1 &= \frac{r}{2} q + \frac{7r}{16} q^2 + \frac{r(8r^2 + 47)}{128} q^3 + \frac{r(272r^2 + 639)}{2048} q^4 + O(q^5), \\ w_2 &= 1 + \frac{q}{4} + \frac{8r^2 + 9}{64} q^2 + \frac{48r^2 + 25}{256} q^3 + \frac{64(6r^2 + 55)r^2 + 1225}{16384} q^4 + O(q^5), \\ \kappa &= \sqrt{q} + \frac{r^2 + 2}{8} q^{3/2} - \frac{r^4 - 24r^2 - 18}{128} q^{5/2} + \frac{r^6 + 10r^4 + 220r^2 + 100}{1024} q^{7/2} + O(q^{9/2}) \end{aligned} \quad (6.13)$$

Expressing the classical energy in terms of  $\mathcal{J}_2$  we get

$$\mathcal{E}_0 = \kappa = \sqrt{2\mathcal{J}_2} \left(1 + \frac{2r^2 + 1}{8} \mathcal{J}_2 - \frac{4r^4 - 28r^2 - 3}{128} \mathcal{J}_2^2 + \frac{8r^6 - 52r^4 + 94r^2 + 1}{1024} \mathcal{J}_2^3 + \dots\right). \quad (6.14)$$

Another option is

$$\mathcal{J}_2 \rightarrow 0, \quad s = \frac{\mathcal{J}_1^2}{\mathcal{J}_2} = \text{fixed}. \quad (6.15)$$

In this case, expanding in small  $q$ , we find

$$\begin{aligned} \mathcal{J}_2 &= \frac{q}{2} + \frac{2s + 3}{16} q^2 + \frac{2s(s + 10) + 15}{128} q^3 + \frac{4s(17s + 81) + 175}{2048} q^4 + O(q^5), \\ w_1 &= \frac{\sqrt{r}}{\sqrt{2}} \sqrt{q} + \frac{\sqrt{s}(2s + 11)}{16\sqrt{2}} q^{3/2} + \frac{\sqrt{s}(4s(s + 25) + 259)}{512\sqrt{2}} q^{5/2} + O(q^{7/2}), \\ w_2 &= 1 + \frac{s + 1}{4} q + \frac{2s(s + 9) + 9}{64} q^2 + \frac{4s(4s + 17) + 25}{256} q^3 \\ &\quad + \frac{4s(-2s^3 + 352s + 985) + 1225}{16384} q^4 + O(q^5), \\ \kappa &= \sqrt{\frac{s}{2} + 1} q^{1/2} + \frac{s(2s + 11) + 8}{16\sqrt{2}\sqrt{s + 2}} q^{3/2} + \frac{4s^4 29s^3 + 515s^2 + 760s + 288}{512\sqrt{2}(s + 2)^{3/2}} q^{5/2} + O(q^{7/2}). \end{aligned} \quad (6.16)$$

Then the classical energy is (cf. (6.14))

$$\begin{aligned} \mathcal{E}_0 = \kappa &= \sqrt{(2 + s)\mathcal{J}_2} \left(1 + \frac{2s + 1}{4s + 8} \mathcal{J}_2 - \frac{4s^3 + 4s^2 - 14s - 3}{32(s + 2)^2} \mathcal{J}_2^2 \right. \\ &\quad \left. + \frac{8s^5 + 8s^4 - 80s^3 - 135s^2 + 1}{128(s + 2)^3} \mathcal{J}_2^3 + \dots\right). \end{aligned} \quad (6.17)$$

## 6.2 Quadratic fluctuation operators

The bosonic fluctuation Lagrangian near this solution was found in conformal gauge in [12]. In  $AdS_5$  we have one massless mode and four modes with  $M^2 = \kappa^2$  while for the  $S^5$  fluctuations we get

$$\begin{aligned}\tilde{L}_{S^5} = & |\dot{X}|^2 - |X'|^2 - M_X^2 |X|^2 + \frac{1}{2}(\dot{\eta}^2 - \eta'^2 - M_\eta^2 \eta^2) + (Q_1 f_1 + Q_2 f_2) \dot{\eta} \\ & + \frac{1}{2}(\dot{f}_1^2 - f_1'^2 - M_1^2 f_1^2) + \frac{1}{2}(\dot{f}_2^2 - f_2'^2 - M_2^2 f_2^2),\end{aligned}\quad (6.18)$$

$$M_X^2 = 2(\kappa^2 - w_1^2) \frac{\sin^2 \theta}{\sin^2 \theta_0} + 2w_1^2 - \kappa^2, \quad M_\eta^2 = -(\kappa^2 - w_1^2) \frac{\cos 2\theta}{\sin^2 \theta_0}, \quad (6.19)$$

$$M_1^2 = -(\kappa^2 - w_1^2) \left(1 - 2 \frac{\sin^2 \theta}{\sin^2 \theta_0}\right), \quad M_2^2 = -(\kappa^2 - w_1^2) \left(1 + \frac{\cos 2\theta}{\sin^2 \theta_0}\right), \quad (6.20)$$

$$Q_1 = 2w_1 \sin \theta, \quad Q_2 = -2w_2 \cos \theta. \quad (6.21)$$

We observe that when both spins are non-trivial there are three coupled bosonic fluctuations and this makes the exact computation of the fluctuation determinant a non-trivial task.

Below we shall consider a particular case with only one non-zero spin  $\mathcal{J}_2$  (the one corresponding to rotation around c.o.m.). In this case the bosonic fluctuations can be decoupled in the static gauge. We shall thus set<sup>16</sup>

$$w_1 = 0, \quad w \equiv w_2, \quad \mathcal{J}_1 = 0. \quad (6.22)$$

In this case the expansions of the classical energy in (6.14) and (6.17) become the same ( $r = s = 0$ )

$$\mathcal{E}_0 = \kappa = \sqrt{2\mathcal{J}_2} \left(1 + \frac{1}{8}\mathcal{J}_2 + \frac{3}{128}\mathcal{J}_2^2 + \frac{1}{1024}\mathcal{J}_2^3 + \dots\right). \quad (6.23)$$

The quadratic bosonic fluctuation action in the static gauge is found to be (see Appendix C; here  $k = 1, 2, 3, 4$ ;  $i = 1, 2, 3$ )

$$\begin{aligned}\tilde{L} = & \frac{1}{2} \left[ (\partial_\sigma \tilde{\eta}_k)^2 - (\partial_\tau \tilde{\eta}_k)^2 + \kappa^2 \tilde{\eta}_k^2 + (\partial_\sigma \tilde{\psi}_i)^2 - (\partial_\tau \tilde{\psi}_i)^2 + (2w^2 \sin^2 \theta - \kappa^2) \tilde{\psi}_i^2 \right. \\ & \left. + (\partial_\sigma f)^2 - (\partial_\tau f)^2 + f^2 \kappa^2 \left(1 - \frac{2(\kappa^2 - w^2)}{\theta'^2}\right) \right].\end{aligned}\quad (6.24)$$

The fermionic fluctuation operator is (here  $s_1 = \text{sign } \theta'$ , see Appendix C)

$$D_F = s_1 \Gamma_0 \partial_\tau - \Gamma_7 \partial_\sigma + u \Gamma_{078} \Gamma_{1234}, \quad (6.25)$$

with the corresponding squared operator whose determinant gives fermionic contribution to 1-loop energy being

$$D_{F\pm}^2 = \partial_\tau^2 - \partial_\sigma^2 + u^2 \pm u', \quad u \equiv w \sin \theta. \quad (6.26)$$

The UV finiteness in the static gauge is checked as follows (cf. (2.19)): the sum of (mass)<sup>2</sup> terms

$$\begin{aligned}AdS & : 4 \times \kappa^2, \\ S^5 & : 3 \times (2w^2 \sin^2 \theta - \kappa^2), \\ & 1 \times \kappa^2 \left(1 - \frac{2(\kappa^2 - w^2)}{\theta'^2}\right), \\ F & : -8 \times w^2 \sin^2 \theta,\end{aligned}\quad (6.27)$$

<sup>16</sup>Let us note that the condition  $\kappa < w$  implies an upper bound for  $\mathcal{J}_2$ .

gives

$$\frac{2}{\theta'^2}[\theta'^4 - \kappa^2(\kappa^2 - w^2)] = \sqrt{-g} R^{(2)}, \quad (6.28)$$

which is the expected value for UV finiteness in the static gauge (cf. (3.25)).

Since there is no nontrivial dependence of the potentials on  $\tau$ , we may switch to Euclidian time  $\tau \rightarrow i\tau$  and replace  $\partial_\tau \rightarrow i\omega$ . Then the relevant 1-d operators will have the form

$$\mathcal{O} = -\partial_\sigma^2 + M^2(\sigma) + \omega^2, \quad (6.29)$$

and can be put as in [6] in the Lamé form allowing us to compute the determinants in a closed form. For the bosonic fluctuation in (6.24) with mass  $M^2 = 2w^2 \sin^2 \theta - \kappa^2$  we obtain the operator

$$\mathcal{O}_I = w^2 \left[ -\partial_x^2 + 2k^2 \text{sn}^2(x|k^2) - \frac{\kappa^2 - \omega^2}{w^2} \right], \quad k^2 \equiv \frac{\kappa^2}{w^2}, \quad x = w\sigma. \quad (6.30)$$

For the fluctuation with mass  $M^2 = \kappa^2(1 - 2\frac{\kappa^2 - w^2}{\theta'^2})$  we get a similar result

$$\mathcal{O}_{II} = w^2 \left[ -\partial_{\bar{x}}^2 + 2k^2 \text{sn}^2(\bar{x}|k^2) - \frac{\kappa^2 - \omega^2}{w^2} \right], \quad \bar{x} \equiv x + i\mathbb{K}' + \mathbb{K}, \quad (6.31)$$

where  $\mathbb{K}' = \mathbb{K}(q')$ ,  $q' = \sqrt{1-q}$ . The fermionic operator in (6.26) can be written as

$$\mathcal{O}_{III} = -\partial_\sigma^2 + \omega^2 + qw^2 \text{sn}^2(w\sigma|q) \pm w^2 \sqrt{q} \text{cn}(w\sigma|q) \text{dn}(w\sigma|q), \quad (6.32)$$

which can also be put in Lamé form as in [6] for both signs in the potential (here we ignore an irrelevant overall constant factor)

$$\mathcal{O}_{III} = -\partial_x^2 + 2\tilde{q} \text{sn}^2(x|\tilde{q}) + \bar{\omega}^2, \quad (6.33)$$

where

$$x = \begin{cases} \frac{\tilde{\mathbb{K}}}{\pi} \sigma, & \text{for } + \text{ sign} \\ \frac{\tilde{\mathbb{K}}}{\pi} \sigma + \tilde{\mathbb{K}}, & \text{for } - \text{ sign} \end{cases}, \quad \tilde{q} = \frac{4\sqrt{q}}{(1+\sqrt{q})^2}, \quad \bar{\omega}^2 = \left(\frac{\pi\omega}{\tilde{\mathbb{K}}}\right)^2 + \tilde{q}, \quad \tilde{\mathbb{K}} \equiv \mathbb{K}(\tilde{q}). \quad (6.34)$$

### 6.3 Short string expansion of the one-loop energy

As in the case of the folded spinning string in  $AdS_3$  the one loop correction to the energy is given simply by

$$E_1 = \frac{1}{\kappa} E_{2d}, \quad E_{2d} = \frac{\Gamma_1}{\mathcal{T}}, \quad (6.35)$$

where  $\Gamma_1$  is the one-loop Euclidean effective action for all fluctuations in the static gauge and  $\mathcal{T}$  is an arbitrary infinite time interval. Below we will consider the expansion of  $E_1$  in the short string (small spin) limit, i.e. expand the determinants in  $\Gamma_1$  in the limit  $q \rightarrow 0$  (cf. (6.11)). We shall not use exact expressions for the determinants as in [6] but rather apply direct perturbation theory in small  $q$ . For small  $q$ , the 1d fluctuation operators are ( $\kappa = \sqrt{q} + \dots$ )

$$\begin{aligned} \mathcal{O}_{AdS_5} &= -\partial_\sigma^2 + \omega^2 + q + \dots, \\ \mathcal{O}_{S^5} &= \mathcal{O}_{I,II} = -\partial_\sigma^2 + \omega^2 + q(2\sin^2 \sigma - 1) + \dots, \\ \mathcal{O}_F &= \mathcal{O}_{III} = -\partial_\sigma^2 + \omega^2 + q\sin^2 \sigma \pm \sqrt{q} \cos \sigma + \dots \end{aligned} \quad (6.36)$$

Using the standard “quantum mechanical” perturbation theory in the basis  $\langle \sigma | n \rangle = \frac{1}{\sqrt{2\pi}} e^{in\sigma}$  we find the leading terms in the spectra<sup>17</sup>

$$\begin{aligned}\Omega^2 &= \omega^2 + \omega_n^2, & \omega_{n,AdS_5}^2 &= n^2 + q + \dots, \\ \omega_{n,S^5}^2 &= n^2 + 0 + \dots & |n| \neq 1, & \omega_{\pm 1,S^5}^2 = 1 \pm \frac{q}{2} + \dots, \\ \omega_{n,F}^2 &= n^2 + q \left( \frac{1}{2} + \frac{1}{2(4n^2 - 1)} \right) + \dots.\end{aligned}\tag{6.37}$$

Here the fermionic contribution is found by combining the first-order perturbation term  $\langle \sin^2 \sigma \rangle = \frac{1}{2}$  with the second-order one order piece

$$\sum_{m \neq n} \frac{|\langle n | \sqrt{q} \cos \sigma | m \rangle|^2}{n^2 - m^2} = \frac{q}{2} \frac{1}{4n^2 - 1}.\tag{6.38}$$

The resulting expression for the 1-loop effective action or 2d energy is then

$$\begin{aligned}E_{2d} &= \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \sum_{n=-\infty}^{\infty} \log \frac{(n^2 + \omega^2 + \omega_{n,S^5}^2)^4 (n^2 + \omega^2 + \omega_{n,AdS_5}^2)^4}{(n^2 + \omega^2 + \omega_{n,F}^2)^8} \\ &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( 4\omega_{n,AdS_5} + 4\omega_{n,S^5} - 8\omega_{n,F} \right).\end{aligned}\tag{6.39}$$

The exact 0-mode ( $n = 0$ ) contribution here is  $2\kappa$ . Summing up the  $n \neq 0$  contributions gives

$$E_{2d} = 2\sqrt{q} + q(2 - 4 \log 2) + O(q^2).\tag{6.40}$$

This computation can be extended to higher orders in  $q$ . We find for the expansion of the operators

$$\begin{aligned}\mathcal{O}_{AdS_5} &= -\partial_\sigma^2 + \omega^2 + q + \frac{q^2}{2} + \frac{11}{32}q^3 + \dots, \\ \mathcal{O}_{S^5} &= -\partial_\sigma^2 + \omega^2 - q \cos 2\sigma + \frac{q^2}{2}[-1 + (3 + \cos 2\sigma) \sin^2 \sigma] \\ &\quad + \frac{q^3}{256}(32 - 85 \cos 2\sigma - 32 \cos 4\sigma - 3 \cos 6\sigma) + \dots, \\ \mathcal{O}_F &= -\partial_\sigma^2 + \omega^2 \pm q^{1/2} \cos \sigma + q \sin^2 \sigma \pm \frac{q^{3/2}}{16}(5 \cos \sigma + 3 \cos 3\sigma) \\ &\quad + \frac{q^2}{4}(3 + \cos 2\sigma) \sin^2 \sigma + \frac{q^{5/2}}{256}(47 \cos \sigma + 36 \cos 3\sigma + 5 \cos 5\sigma) \\ &\quad + \frac{q^3}{128}(79 + 38 \cos 2\sigma + 3 \cos 4\sigma) \sin^2 \sigma + \dots,\end{aligned}\tag{6.41}$$

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<sup>17</sup>Here the  $S^5$  mode at  $n = 1$  requires diagonalization of a  $2 \times 2$  matrix, or the change to the basis  $\sin \sigma, \cos \sigma$ . We formally associate the two corresponding eigenvalues to  $n = \pm 1$ .

and the perturbation theory then gives the expansion of the spectra

$$\begin{aligned}
\omega_{n,AdS_5}^2 &= n^2 + q + \frac{q^2}{2} + \frac{11}{32}q^3 + \dots, \\
\omega_{n,S^5}^2 &= n^2 + 0 \cdot q + \frac{n^2}{8(n^2-1)}(q^2 + q^3) + \dots, \quad |n| \neq 1, \\
\omega_{1,S^5}^2 &= 1 + \frac{1}{2}q + \frac{11}{32}q^2 + \frac{17}{64}q^3 + \dots, \quad \omega_{-1,S^5}^2 = 1 - \frac{1}{2}q - \frac{5}{32}q^2 - \frac{5}{64}q^3 + \dots, \\
\omega_{n,F}^2 &= n^2 + q \frac{2n^2}{4n^2-1} + q^2 \frac{n^2(-5-32n^2+80n^4)}{4(-1+4n^2)^3} \\
&\quad + q^3 \frac{n^2 [2(960n^6 - 816n^4 + 60n^2 + 79)n^2 + 3]}{8(4n^2-1)^5} + \dots.
\end{aligned} \tag{6.42}$$

Treating separately the  $n = 0, -1, 1$  and  $|n| \geq 2$  terms in (6.39) we find for the 2d energy

$$E_{2d} = 2\kappa + q(2 - 4 \log 2) + q^2 \left( \frac{5}{8} - \frac{5}{2} \log 2 + \frac{3}{8} \zeta_3 \right) + q^3 \left( \frac{3}{8} - \frac{15}{8} \log 2 + \frac{45}{64} \zeta_3 - \frac{15}{64} \zeta_5 \right) + \dots \tag{6.43}$$

Dividing by  $\kappa$  and expressing everything in terms of  $\mathcal{J}_2$ , we arrive at

$$\begin{aligned}
E_1 &= 2 + \sqrt{2\mathcal{J}_2} \left[ 2 - 4 \log 2 - \mathcal{J}_2 \left( \frac{1}{2} + \frac{3}{2} \log 2 - \frac{3}{4} \zeta_3 \right) \right. \\
&\quad \left. + \mathcal{J}_2^2 \left( \frac{1}{64} - \frac{15}{32} \log 2 + \frac{51}{32} \zeta_3 - \frac{15}{16} \zeta_5 \right) + \dots \right].
\end{aligned} \tag{6.44}$$

## 7 Summary and concluding remarks

In this paper we continued the investigation [6] of the exact structure of one-loop correction to energy of an important class of classical string solutions in  $AdS_5 \times S^5$  expressed in terms simple elliptic functions. This elliptic class is next in complexity to the simplest rational class [5, 31] for which the classical solutions are expressed in terms of linear or trigonometric functions of world-sheet coordinates and thus the quadratic fluctuation operators can be put into the form where their coefficients are constant and thus their spectrum can be easily found.

The elliptic solution considered in [6] was the folded spinning string in  $AdS_5$  for which it was shown that the quadratic fluctuation operators can be put into the standard single-gap Lamé form; that allows one to compute the corresponding determinants and thus the one-loop correction to the string energy exactly for any value of semiclassical spin parameter  $\mathcal{S}$ . Here we have demonstrated that the same is true also for other basic elliptic solutions: the pulsating string in  $\mathbb{R} \times S^2$ , the pulsating string in  $AdS_3$  and the folded spinning string in  $\mathbb{R} \times S^2$ . In all of these cases where there is only one charge/adiabatic invariant besides the energy, namely, an oscillator number or spin (in  $S^5$  or  $AdS_5$ ), the fluctuation operators can be decoupled and put into a single-gap Lamé type form. In fact, there is an explicit analytic continuation between the pulsating string and folded string cases. For example, in the  $\mathbb{R} \times S^2$  case, the mapping  $i\omega \leftrightarrow \sqrt{m^2 - \kappa^2}$  maps the classical conserved quantities into one another:  $\omega \mathcal{J}_2 \leftrightarrow i\mathcal{N}$ , as can be seen from (2.7) and (6.10) after using some elliptic function identities. Moreover, the fluctuation operators also map into one another, with the identification:  $\omega \sigma \leftrightarrow \sqrt{m^2 - \kappa^2} \tau + \mathbb{K} \left( \frac{\kappa^2}{\kappa^2 - m^2} \right)$ . While this does not directly imply the equivalence of the corresponding expressions for the one-loop energies, this relation is quite intriguing and is worth further study.

We have found, in particular, the expansion of the one-loop energies in the limit of small values of the semiclassical parameters corresponding to small size of the string. This is equivalent to the “near-flat” approximation when the string probes only small region of  $AdS_5 \times S^5$  so that its energy should start with the standard flat-space form plus corrections due to curvature. As was argued in [3, 4] this “short-string” limit (in which the finite-size effects of the compact  $\sigma \equiv \sigma + 2\pi$  string direction are all taken into account) may shed light on the structure of strong-coupling corrections to dimensions of “short” dual gauge theory operators for which the “wrapping” contributions are important.<sup>18</sup>

The semiclassical approximation is based on assumption that  $\sqrt{\lambda} \gg 1$  with semiclassical parameters like  $\mathcal{S} = \frac{S}{\sqrt{\lambda}}$ ,  $\mathcal{J} = \frac{J}{\sqrt{\lambda}}$  or  $\mathcal{N} = \frac{N}{\sqrt{\lambda}}$  fixed, so that  $S, J$  or  $N$  are formally large. Still, taking the “short-string” limit in which  $\mathcal{S}, \mathcal{J}, \mathcal{N} \rightarrow 0$  one may conjecture that if that limit “commutes” with large the  $\sqrt{\lambda}$  limit it may shed light on the form of the quantum string energies with fixed (e.g., small) values of the spins and oscillation numbers ( $S, J, N$ ). While this conjecture is hard to justify at the moment, the study of the “short-string” limit appears to provide some qualitative information on the structure of the large tension expansion of quantum string energies or strong-coupling expansion of dimensions of dual gauge-theory operators.

## 7.1 Results

Below we summarize the results for the “short-string” (small spin or oscillation number) expansion of the classical  $E_0$  and one-loop  $E_1$  energies of the four basic elliptic  $AdS_5 \times S^5$  solutions analysed in [6] and here: folded spinning strings in  $\mathbb{R} \times S^2$  and  $AdS_3$ , and pulsating circular strings in  $\mathbb{R} \times S^2$  and  $AdS_3$ . We consider the case of minimal winding number  $m = 1$ . We recall our notation:  $E = E_0 + E_1 + \dots$ ,  $E_0 = \sqrt{\lambda} \mathcal{E}_0$ ,  $E_1 = \mathcal{E}_1$ . Also, the non-zero spin in  $S^2$  is  $J_2 \equiv J$ .

Folded spinning string in  $\mathbb{R} \times S^2$

$$\begin{aligned}\mathcal{E}_0 &= \sqrt{2\mathcal{J}} \left( 1 + \frac{1}{8}\mathcal{J} + \frac{3}{128}\mathcal{J}^2 + \dots \right), \\ \mathcal{E}_1 &= 2 + \sqrt{2\mathcal{J}} \left[ 2 - 4 \log 2 + \left( -\frac{1}{2} - \frac{3}{2} \log 2 + \frac{3}{4}\zeta_3 \right) \mathcal{J} \right. \\ &\quad \left. + \left( \frac{1}{64} - \frac{15}{32} \log 2 + \frac{51}{32}\zeta_3 - \frac{15}{16}\zeta_5 \right) \mathcal{J}^2 + \dots \right], \\ E &= \sqrt{2J\sqrt{\lambda}} \left( 1 + \frac{\frac{1}{8}J + 2 - 4 \log 2}{\sqrt{\lambda}} + \dots \right) + 2 + \dots\end{aligned}\tag{7.1}$$

Folded spinning string in  $AdS_3$

$$\begin{aligned}\mathcal{E}_0 &= \sqrt{2\mathcal{S}} \left( 1 + \frac{3}{8}\mathcal{S} - \frac{21}{128}\mathcal{S}^2 + \dots \right), \\ \mathcal{E}_1 &= 1 + \sqrt{2\mathcal{S}} \left[ \frac{3}{2} - 4 \log 2 + \left( -\frac{23}{16} + \frac{3}{2} \log 2 + \frac{3}{4}\zeta_3 \right) \mathcal{S} \right. \\ &\quad \left. + \left( \frac{689}{256} - \frac{63}{32} \log 2 - \frac{15}{32}\zeta_3 - \frac{15}{16}\zeta_5 \right) \mathcal{S}^2 + \dots \right], \\ E &= \sqrt{2S\sqrt{\lambda}} \left( 1 + \frac{\frac{3}{8}S + \frac{3}{2} - 4 \log 2}{\sqrt{\lambda}} + \dots \right) + 1 + \dots\end{aligned}\tag{7.2}$$

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<sup>18</sup>While the result of [2] guarantees that the strong coupling expansion of TBA in similar semiclassical limit should reproduce the full string semiclassical one-loop correction, that was shown only in the  $sl(2)$  sector in the limit when the orbital momentum  $\mathcal{J}$  in  $S^5$  is non-zero. The limit  $\mathcal{J} \rightarrow 0$  may be subtle, and therefore the explicit string-theory results provide important data points.

### Pulsating string in $\mathbb{R} \times S^2$

$$\begin{aligned}
\mathcal{E}_0 &= \sqrt{2\mathcal{N}} \left( 1 - \frac{1}{8}\mathcal{N} - \frac{5}{128}\mathcal{N}^2 + \dots \right), \\
\mathcal{E}_1 &= 2 + \sqrt{2\mathcal{N}} \left[ 1 - 4 \log 2 + \left( \frac{1}{8} + \frac{3}{2} \log 2 + \frac{3}{4}\zeta_3 \right) \mathcal{N} \right. \\
&\quad \left. + \left( \frac{11}{128} + \frac{25}{32} \log 2 - \frac{135}{32}\zeta_3 - \frac{15}{16}\zeta_5 \right) \mathcal{N}^2 + \dots \right], \\
E &= \sqrt{2N\sqrt{\lambda}} \left( 1 + \frac{-\frac{1}{8}N + 1 - 4 \log 2}{\sqrt{\lambda}} + \dots \right) + 2 + \dots
\end{aligned} \tag{7.3}$$

### Pulsating string in $AdS_3$

$$\begin{aligned}
\mathcal{E}_0 &= \sqrt{2\mathcal{N}} \left( 1 + \frac{5}{8}\mathcal{N} - \frac{77}{128}\mathcal{N}^2 + \dots \right), \\
\mathcal{E}_1 &= 1 + \sqrt{2\mathcal{N}} \left[ \frac{5}{2} - 4 \log 2 + \left( -\frac{37}{8} + \frac{5}{2} \log 2 + \frac{3}{4}\zeta_3 \right) \mathcal{N} \right. \\
&\quad \left. + \left( \frac{3915}{256} - \frac{231}{32} \log 2 - \frac{117}{32}\zeta_3 - \frac{15}{16}\zeta_5 \right) \mathcal{N}^2 + \dots \right], \\
E &= \sqrt{2N\sqrt{\lambda}} \left( 1 + \frac{\frac{5}{8}N + \frac{5}{2} - 4 \log 2}{\sqrt{\lambda}} + \dots \right) + 1 + \dots,
\end{aligned} \tag{7.4}$$

We observe a remarkable universality of the small charge expansion of the energy of all four elliptic solutions.<sup>19</sup> In particular, the leading terms with transcendental coefficients ( $\log 2$ ,  $\zeta_3$ ,  $\zeta_5$ , ...) happen to have the same form. Compared to similar expansions for rational rigid spinning string solutions discussed in [4] we notice the presence of the  $\log 2$  term already in the leading one-loop coefficient which was absent in the rational case.<sup>20</sup>

## 7.2 Interpolation to finite quantum numbers: energies of strings corresponding to first excited string level

The semiclassical approximation discussed above was based on assumption that one takes  $\sqrt{\lambda} \gg 1$  for fixed  $\mathcal{Q} = \frac{Q}{\sqrt{\lambda}}$  with  $Q = (N, S, J, \dots)$  and then expands in  $\mathcal{Q} \rightarrow 0$ . This still means that  $Q \gg 1$ . As was argued in [4], if one assumes that the resulting expressions for string energy can be formally interpolated to finite values of  $Q$  they should then describe leading corrections to energies of the corresponding quantum string states. In particular, one may consider the analogs of states at the first excited string level which should correspond to members of the Konishi multiplet [35, 4] (if this is the case their energies should differ only by  $\lambda$ -independent half-integer constants).

Interpolation from semiclassical expressions for  $E$  like given above, i.e.  $E = \sqrt{2Q\sqrt{\lambda}}(1 + \frac{aQ+b}{\sqrt{\lambda}} + \dots)$  valid for  $\sqrt{\lambda} \gg 1$  and fixed  $\mathcal{Q} \ll 1$ , i.e.  $Q \gg 1$ , to quantum string energies with finite  $Q$  is, of course, potentially ambiguous. One requirement is that one should match the corresponding flat-space expressions. In [4] this ambiguity was fixed by shifting  $Q \rightarrow Q - 2$  everywhere in the expression

<sup>19</sup>One may wonder if the 1-loop expressions we found are scheme-dependent. The choice of scheme preserving all relevant symmetries in computations with GS action is a subtle issue that deserves further study (see [29] for a discussion). Here as in several previous papers we assumed that bosonic and fermionic contributions are first added together and then the (finite) sum over modes is performed.

<sup>20</sup>The universality of the  $\log 2$  coefficient suggests that maybe it can be absorbed into a redefinition of  $\lambda$  (cf. cusp anomaly case [33]). Indeed, a simple shift of  $\sqrt{\lambda}$  by  $4 \ln 2$  removes the leading  $\ln 2$  terms in the 1-loop correction, but it does not remove  $\ln 2$  coefficients in subleading terms so we are not sure if that this shift may have a deeper meaning.



for the energy, i.e.  $E(\sqrt{\lambda}, Q) \rightarrow E(\sqrt{\lambda}, Q - 2)$ . An alternative recipe that we shall consider here is to do this shift only under the square root

$$E = \sqrt{2(Q-2)\sqrt{\lambda}} \left( 1 + \frac{aQ+b}{\sqrt{\lambda}} + \dots \right). \quad (7.5)$$

One may think that this is suggested by the structure of the solution of the the marginality condition for the corresponding vertex operator which looks like  $2 = Q - \frac{1}{2\sqrt{\lambda}}[E(E-4) - aQ(Q+b) + \dots] + \dots$ , see [4].<sup>21</sup> Then to get the energies of states on the first excited string level we should start with (7.5) and set  $Q = (N, J, S) = 4$  (for states on the first excited string level the corresponding vertex operators should contain factors like  $(\partial x \bar{\partial} x)^{Q/2} = (\partial x \bar{\partial} x)^2$ , etc.).

Using the above results (7.1)–(7.4) we then get

$$E_{\text{folded } \mathbb{R} \times S^2} = 2\sqrt[4]{\lambda} \left( 1 + \frac{\frac{5}{2} - 4 \log 2}{\sqrt{\lambda}} + \dots \right) + 2, \quad (7.6)$$

$$E_{\text{folded } \text{AdS}_3} = 2\sqrt[4]{\lambda} \left( 1 + \frac{3 - 4 \log 2}{\sqrt{\lambda}} + \dots \right) + 1, \quad (7.7)$$

$$E_{\text{pulsating } \mathbb{R} \times S^2} = 2\sqrt[4]{\lambda} \left( 1 + \frac{\frac{1}{2} - 4 \log 2}{\sqrt{\lambda}} + \dots \right) + 2, \quad (7.8)$$

$$E_{\text{pulsating } \text{AdS}_3} = 2\sqrt[4]{\lambda} \left( 1 + \frac{5 - 4 \log 2}{\sqrt{\lambda}} + \dots \right) + 1. \quad (7.9)$$

The difference of the coefficients of the first subleading term here and for the rational solutions in [4] may be due to the fact that these states are not actually in the same supermultiplet so dimensions need not be related just by an integer number shift.<sup>22</sup> An alternative is that the semiclassical expressions cannot be actually interpolated to fixed values of quantum numbers. That issue remains to be clarified; still, the similarity of the above expressions and those in [4] for energies of “small” strings suggest that they do model quantum string energies, i.e. are not very much off the mark.

### Note added

In discussing one-loop corrections for pulsating strings in section 5 we have tacitly assumed that the fermions in (2.14) or (A.15) with the angular (“polar”) choice of global coordinates like in (A.16) are periodic in  $\sigma$  for any value of the winding number  $m$ . As was pointed out to us by Victor Mikhaylov after the first version of this paper appeared on the arXiv, this may be unnatural in view of the discussion in [26]: the fermions should be periodic for any  $m$  in “cartesian” coordinates but that implies that they should be antiperiodic for  $m$ =odd in “polar” coordinates [27]. For example, in flat space, changing coordinates from cartesian to polar seems to require rotation of the GS fermions  $\theta$  (target-space spinors) by an angle  $\phi$ , so that for a circular solution with  $\phi = m\sigma$  starting with  $\theta(\sigma + 2\pi) = \theta(\sigma)$  one should end up with  $\tilde{\theta}(\sigma + 2\pi) = (-1)^m \tilde{\theta}(\sigma)$ .<sup>23</sup> We do not, however, find this reasoning convincing since in curved space (or in general coordinates) the target-space spinors

<sup>21</sup>At the same time, the recipe of [4], i.e.  $Q \rightarrow Q - 2$ , may be motivated by the requirement that not only the leading term but also  $Q$ -dependent corrections should vanish for the BPS ground-state cases with  $Q = 2$ .

<sup>22</sup>For example, the folded string in  $AdS_5$  without orbital momentum in  $S^5$  may be dual to an operator built out field strengths like  $\text{Tr}(FD^S F)$  that mixes with other similar operators and is not in the Konishi multiplet. In general, the question of identification of states in semiclassical expansion is subtle as finite values of  $S^5$  orbital momentum cannot be resolved, so one cannot a priori distinguish between a state dual to  $\text{Tr}(FD^S F)$  and a state dual to  $\text{Tr}(\Phi D^S \Phi)$ .

<sup>23</sup>As for the folded string in  $AdS_3$  or  $\mathbb{R} \times S^2$ , there is no obvious reason to change from periodic to antiperiodic fermions

do not transform under diffeomorphisms but rotate under local Lorentz frame transformations (with the tangent-space metric and Dirac  $\Gamma$ -matrices being the standard Minkowski ones for any choice of the coordinate labels).<sup>24</sup> To clarify this further, in Appendix F we discuss the fermionic kinetic term in the light-cone gauge adding also angular momentum in  $S^5$  that allows one to interpolate to the BMN limit. In the case of pulsating string in flat space we explicitly show that the issue of periodicity/antiperiodicity of fermions is indeed a gauge/coordinate artifact.

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## Appendices

### A Fluctuation Lagrangian for pulsating solution in $\mathbb{R} \times S^2$

Here we present details of computation of fluctuation Lagrangian for pulsating string in  $\mathbb{R} \times S^2$ .

In conformal gauge the  $AdS_5$  part of fluctuation Lagrangian contains one massless mode (fluctuation of  $t$ ) and 4 massive modes with mass  $\kappa$ . The  $S^5$  part of Lagrangian written in terms of complex combinations of 6 embedding coordinates is

$$L_S = -\frac{1}{2} \partial_a Z_i \partial^a Z_i^* + \frac{\Lambda}{2} (Z_i Z_i^* - 1), \quad (\text{A.1})$$

where for pulsating solution

$$Z_1 = \cos \psi(\tau), \quad Z_2 = \sin \psi(\tau) e^{im\sigma}, \quad Z_3 = 0, \quad \Lambda = 2m^2 \sin^2 \psi - \kappa^2. \quad (\text{A.2})$$

The fluctuations of  $Z_i$  satisfying  $Z_i \tilde{Z}_i^* + Z_i^* \tilde{Z}_i = 0$  contain two massive modes  $\tilde{Z}_3$  with mass  $M_3^2 = -\Lambda = \kappa^2 - 2m^2 \sin^2 \psi$  and two coupled fluctuations

$$Z_1 = \cos \psi + g_1 + i z, \quad Z_2 = (\sin \psi + g_2 + i \xi) e^{im\sigma}, \quad \zeta \equiv g_1 \cos \psi + g_2 \sin \psi = 0. \quad (\text{A.3})$$

Introducing  $\eta = g_2 \cos \psi - g_1 \sin \psi$  orthogonal to  $\zeta$  we end up with the following fluctuation Lagrangian for the 3 remaining modes<sup>25</sup>

$$\begin{aligned} \tilde{L} = & \frac{1}{2}(\dot{\eta}^2 - \eta'^2 - M_\eta^2 \eta^2) + \frac{1}{2}(\dot{z}^2 - z'^2 - M^2 z^2) + \frac{1}{2}(\dot{\xi}^2 - \xi'^2 - M_\xi^2 \xi^2) + \\ & + m \cos \psi (\xi \eta' - \xi' \eta). \end{aligned} \quad (\text{A.4})$$

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(the corresponding rotation from cartesian to polar coordinates is  $\tau$ -dependent). Let us note, however, a somewhat special “singular” nature of the folded string which may be considered as a special case of a spiky string [28] which does encircle the origin like a pulsating string.

<sup>24</sup>It is true, of course, that there is only one antiperiodic spin structure on the disc which is familiar in the open NSR string case but in the closed string case where the image of the world sheet in the target space should be a cylinder the situation is different. For example, in the static gauge where  $t = \kappa\tau$ ,  $\phi = m\sigma$  become arguments of  $\theta$  the latter should still be defined on a cylinder.

<sup>25</sup>We thank I. Park for pointing out to us some typos in this Appendix.

To decouple  $\eta$  and  $\xi$  fluctuations we may use the linearized Virasoro constraints (see [6] for a similar discussion)

$$m \sin \psi (m \cos \psi \eta + \xi') - \kappa \dot{\beta} + \dot{\psi} \dot{\eta} = 0, \quad m \sin \psi \dot{\xi}_2 - m \cos \psi \dot{\psi} \xi - \kappa \beta' + \dot{\psi} \eta' = 0, \quad (\text{A.5})$$

where  $\beta$  is the massless mode from  $AdS_5$ . Using the equations of motion for  $\eta$  and  $\xi$  fluctuations written for  $\sim e^{in\sigma}$  Fourier mode in  $\sigma$

$$-\ddot{\eta} - (n^2 + M_\eta^2)\eta - 2imn \cos \psi \xi = 0, \quad (\text{A.6})$$

$$-\ddot{\xi} - (n^2 + M_\xi^2)\xi + 2imn \cos \psi \eta = 0. \quad (\text{A.7})$$

we get  $\eta = \frac{\ddot{\xi} + (n^2 + M_\xi^2)\xi}{2imn \cos \psi}$ , and thus obtain the following equation for  $\xi$

$$\frac{1}{\cos \psi}(\partial_\tau^2 + n^2 + M_\eta^2)\frac{1}{\cos \psi}(\partial_\tau^2 + n^2 + M_\xi^2)\xi - 4m^2 n^2 \xi = 0. \quad (\text{A.8})$$

This equation can be written in a factorized form

$$\frac{1}{\sin \psi \cos \psi}(\partial_\tau^2 + \mu^2)\frac{\sin^2 \psi}{\cos \psi}(\partial_\tau^2 + n^2)\frac{\xi}{\sin \psi} = 0, \quad \mu^2 = n^2 + \kappa^2\left(1 - \frac{2}{\sin^2 \psi}\right). \quad (\text{A.9})$$

Thus we end up with two decoupled modes – a massless mode and a mode with mass  $M^2 = \kappa^2(1 - \frac{2}{\sin^2 \psi})$ .

The same decoupling happens directly if we start with the Nambu action and use the static gauge on the fluctuations of  $t$  and  $\phi$ . If we parametrise the metric as in [29] ( $k = 1, 2, 3, 4$ )

$$\begin{aligned} ds^2 = & -\left(\frac{1 + \frac{1}{4}\eta^2}{1 - \frac{1}{4}\eta^2}\right)^2 dt^2 + \frac{d\eta_k d\eta_k}{(1 - \frac{1}{4}\eta^2)^2} \\ & + \frac{dx^2 + dy^2 - (xdy - ydx)^2}{1 - x^2 - y^2} + (1 - x^2 - y^2)(d\psi^2 + \cos^2 \psi d\varphi^2 + \sin^2 \psi d\phi^2) \end{aligned} \quad (\text{A.10})$$

so that pulsating solution is

$$t = \kappa\tau, \quad \eta_k = 0, \quad x = y = 0, \quad \psi = \psi(\tau), \quad \varphi = 0, \quad \phi = m\sigma \quad (\text{A.11})$$

Expanding the Nambu action with  $\tilde{t} = 0$  and  $\tilde{\phi} = 0$  we get

$$\begin{aligned} \tilde{S} = & \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[ (\partial_\sigma \tilde{\eta}_k)^2 - (\partial_\tau \tilde{\eta}_k)^2 + \kappa^2 \tilde{\eta}_k^2 + (\partial_\sigma \tilde{x})^2 - (\partial_\tau \tilde{x})^2 + (\kappa^2 - 2m^2 \sin^2 \psi) \tilde{x}^2 \right. \\ & + (\partial_\sigma \tilde{y})^2 - (\partial_\tau \tilde{y})^2 + (\kappa^2 - 2m^2 \sin^2 \psi) \tilde{y}^2 + \cos^2 \psi [(\partial_\sigma \tilde{\varphi})^2 - (\partial_\tau \tilde{\varphi})^2] \\ & \left. + \frac{\kappa^2}{m^2} \frac{1}{\sin^2 \psi} [(\partial_\sigma \tilde{\psi})^2 - (\partial_\tau \tilde{\psi})^2] - m^2 \sin^2 \psi \tilde{\psi}^2 - \partial_\tau \psi \tilde{\psi} \partial_\tau \tilde{\psi} \cot \psi \right]. \end{aligned} \quad (\text{A.12})$$

With the field redefinitions

$$\cos \psi \tilde{\varphi} = \xi, \quad \frac{\kappa}{m \sin \psi} \tilde{\psi} = g \quad (\text{A.13})$$

we finish with

$$\begin{aligned} \tilde{S} = & \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[ (\partial_\sigma \tilde{\eta}_k)^2 - (\partial_\tau \tilde{\eta}_k)^2 + \kappa^2 \tilde{\eta}_k^2 + (\partial_\sigma \tilde{x})^2 - (\partial_\tau \tilde{x})^2 + (\kappa^2 - 2m^2 \sin^2 \psi) \tilde{x}^2 \right. \\ & + (\partial_\sigma \tilde{y})^2 - (\partial_\tau \tilde{y})^2 + (\kappa^2 - 2m^2 \sin^2 \psi) \tilde{y}^2 + (\partial_\sigma \xi)^2 - (\partial_\tau \xi)^2 + (\kappa^2 - 2m^2 \sin^2 \psi) \xi^2 \\ & \left. + (\partial_\sigma g)^2 - (\partial_\tau g)^2 + \kappa^2 \left(1 - \frac{2}{\sin^2 \psi}\right) g^2 \right] \end{aligned} \quad (\text{A.14})$$

To get the fermionic part of the fluctuation Lagrangian we start with the standard form of the action (see, e.g., [15, 14])

$$\mathcal{L}_F = -2i\bar{\vartheta} \left( -\rho^a D_a - \frac{i}{2} \varepsilon^{ab} \rho_a \Gamma_* \rho_b \right) \vartheta. \quad (\text{A.15})$$

Using the standard choice of global angular  $AdS_5 \times S^5$  coordinates we have

$$\begin{array}{c|cccccccccc} \mu & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline X^\mu & \kappa\tau & 0 & 0 & 0 & 0 & \pi/2 & 0 & \psi(\tau) & 0 & m\sigma \end{array} \quad (\text{A.16})$$

The non zero vielbein and spin connection components are

$$E_0^0 = E_1^1 = E_5^5 = E_7^7 = 1, \quad E_8^8 = \cos \psi, \quad E_9^9 = \sin \psi, \quad (\text{A.17})$$

$$\omega_2^{12} = -1, \quad \omega_3^{13} = -1, \quad \omega_4^{24} = -1, \quad \omega_6^{56} = 1, \quad \omega_8^{78} = \sin \psi, \quad \omega_9^{79} = -\cos \psi, \quad (\text{A.18})$$

so that

$$D_\tau = \partial_\tau, \quad D_\sigma = \partial_\sigma - \frac{1}{2} m \cos \psi \Gamma_{79}, \quad \rho_\tau = \kappa \Gamma_0 + \dot{\psi} \Gamma_7, \quad \rho_\sigma = m \sin \psi \Gamma_9 \quad (\text{A.19})$$

and finally  $\mathcal{L}_F = -2i\bar{\vartheta} D_F \vartheta$ , with

$$D_F = (\kappa \Gamma_0 + \dot{\psi} \Gamma_7) \partial_\tau - m \sin \psi \Gamma_9 \partial_\sigma - \frac{m^2}{2} \sin \psi \cos \psi \Gamma_7 + m \sin \psi \dot{\psi} \Gamma_{07} \Gamma_{12349} \quad (\text{A.20})$$

Performing a Lorentz rotation

$$\Gamma_0(s) = e^{\frac{1}{2}s\Gamma_{07}} \Gamma_0 e^{-\frac{1}{2}s\Gamma_{07}}, \quad \Gamma_7(s) = e^{\frac{1}{2}s\Gamma_{07}} \Gamma_7 e^{-\frac{1}{2}s\Gamma_{07}}. \quad (\text{A.21})$$

with  $\sinh s = -\frac{\dot{\psi}}{\sqrt{\kappa^2 - \dot{\psi}^2}}$ ,  $\cosh s = \frac{\kappa}{\sqrt{\kappa^2 - \dot{\psi}^2}}$ , we obtain

$$D'_F = e^{\frac{1}{2}s\Gamma_{07}} D_F e^{-\frac{1}{2}s\Gamma_{07}} = m \sin \psi \left[ \Gamma_0 (\partial_\tau - \frac{1}{2} \Gamma_{07} \partial_\tau s) - \Gamma_9 \partial_\sigma \right] - \frac{m}{2} \cos \psi (\kappa \Gamma_7 - \dot{\psi} \Gamma_0) + m \sin \psi \dot{\psi} \Gamma_{07} \Gamma_{12349}$$

Simplifying this we get

$$D'_F = m \sin \psi (\Gamma_0 \partial_\tau - \Gamma_9 \partial_\sigma) + \frac{m}{2} \dot{\psi} \cos \psi \Gamma_0 + m \sin \psi \dot{\psi} \Gamma_{07} \Gamma_{12349} \quad (\text{A.22})$$

Rescaling of the fermions  $\vartheta \rightarrow \vartheta \frac{1}{\sqrt{m \sin \psi}}$ ,  $D''_F = \frac{1}{\sqrt{m \sin \psi}} D'_F \frac{1}{\sqrt{m \sin \psi}}$ , gives

$$D''_F = \Gamma_0 \partial_\tau - \Gamma_9 \partial_\sigma + \dot{\psi} \Gamma_{0123479}. \quad (\text{A.23})$$

Diagonalizing  $\Gamma_{1234} = \pm 1$  we end with

$$D_F = \Gamma_0 \partial_\tau - \Gamma_9 \partial_\sigma + \dot{\psi} \Gamma_{079} \quad (\text{A.24})$$

## B Fluctuation Lagrangian for pulsating solution in $AdS_3$

Here we shall use the following coordinates

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho [d\beta_1^2 + \cos^2 \beta_1 (d\beta_2^2 + \cos^2 \beta_2 d\beta_3^2)] \\ + d\psi_1^2 + \cos^2 \psi_1 [d\psi_2^2 + \cos^2 \psi_2 (d\psi_3^2 + \cos^2 \psi_3 (d\psi_4^2 + \cos^2 \psi_4 d\psi_5^2))] \quad (B.1)$$

The pulsating solution in these coordinates is

$$\beta_1 = \beta_3 = 0, \quad \beta_2 = m\sigma, \quad t = t(\tau), \quad \rho = \rho(\tau), \quad \psi_i = 0 \quad (B.2)$$

Fixing the fluctuations of  $t$  and  $\beta_2$  to zero and expanding the Nambu-Goto action we obtain the following fluctuations Lagrangian for the physical 8 fields

$$\tilde{L} = \frac{1}{2} \left[ \sinh^2 \rho [(\partial_\sigma \tilde{\beta}_1)^2 - (\partial_\tau \tilde{\beta}_1^2) - m^2 \tilde{\beta}_1^2] + \sinh^2 \rho \cos^2 m\sigma [(\partial_\sigma \tilde{\beta}_3)^2 - (\partial_\tau \tilde{\beta}_3^2)] \right. \\ + \frac{4\kappa^2}{m^2} \frac{1}{\sinh^2(2\rho)} [(\partial_\sigma \tilde{\rho})^2 - (\partial_\tau \tilde{\rho})^2] + \left[ \frac{\kappa^2(1 + 2 \cosh 2\rho)}{\cosh^4 \rho} - \frac{\kappa^4}{m^2 \cosh^6 \rho} + m^2 \sinh^2 \rho \right] \tilde{\rho}^2 \\ \left. + \frac{8\kappa^2 - 3m^2 - m^2(4 \cosh 2\rho + \cosh 4\rho)}{4m^2 \sinh \rho \cosh^3 \rho} \partial_\tau \rho \tilde{\rho} \partial_\tau \tilde{\rho} + (\partial_\sigma \tilde{\psi}_i)^2 - (\partial_\tau \tilde{\psi}_i)^2 \right] \quad (B.3)$$

where  $\kappa = \mathcal{E}_0$  is the integration constant in (3.5). After the field redefinitions

$$\tilde{\beta}_3 \cos m\sigma \sinh \rho = \eta, \quad \tilde{\beta}_1 \sinh \rho = \xi, \quad \frac{2\kappa}{m \sinh 2\rho} \tilde{\rho} = \zeta \quad (B.4)$$

the fluctuation Lagrangian becomes (after integration by parts)

$$\tilde{L} = \frac{1}{2} \left[ (\partial_\sigma \tilde{\psi}_i)^2 - (\partial_\tau \tilde{\psi}_i)^2 + (\partial_\sigma \eta)^2 - (\partial_\tau \eta)^2 + 2m^2 \eta^2 \sinh^2 \rho + (\partial_\sigma \xi)^2 - (\partial_\tau \xi)^2 + 2m^2 \xi^2 \sinh^2 \rho \right. \\ \left. + (\partial_\sigma \zeta)^2 - (\partial_\tau \zeta)^2 + \zeta^2 (2m^2 \sinh^2 \rho - \frac{2\kappa^2}{\sinh^2 \rho}) \right] \quad (B.5)$$

To find the fermionic Lagrangian we label directions as

$$\begin{array}{c|ccccc} \mu & 0 & 1 & 2 & 3 & 4 \\ \hline X^\mu & t(\tau) & \rho(\tau) & 0 & m\sigma & 0 \end{array} \quad (B.6)$$

The relevant non zero vielbein and connection components are

$$E_0^0 = \cosh \rho, \quad E_1^1 = 1, \quad E_2^2 = \sinh \rho, \quad E_3^3 = \sinh \rho, \quad (B.7)$$

$$\omega_0^{01} = \sinh \rho, \quad \omega_2^{12} = -\cosh \rho, \quad \omega_3^{13} = -\cosh \rho, \quad \omega_4^{24} = -1, \quad (B.8)$$

so that

$$D_\tau = \partial_\tau + \frac{1}{2} \kappa \frac{\sinh \rho}{\cosh^2 \rho} \Gamma_{01}, \quad D_\sigma = \partial_\sigma - \frac{m}{2} \cosh \rho \Gamma_{13} \quad (B.9)$$

$$\rho_\tau = \cosh \rho \dot{t} \Gamma_0 + \dot{\rho} \Gamma_1 = \frac{\kappa}{\cosh \rho} \Gamma_0 + \dot{\rho} \Gamma_1, \quad \rho_\sigma = m \sinh \rho \Gamma_3. \quad (B.10)$$

The fermionic operator is then

$$\begin{aligned}
D_F &= -\rho^a D_a - i \rho_\tau \Gamma_* \rho_\sigma = -\rho^a D_a + \rho_\tau \rho_\sigma \Gamma_{01234} \\
&= (\dot{t} \cosh \rho \Gamma_0 + \dot{\rho} \Gamma_1) \partial_\tau - m \sinh \rho \Gamma_3 \partial_\sigma - \frac{m^2}{2} \sinh \rho \cosh \rho \Gamma_1 \\
&\quad + m \sinh \rho (\dot{t} \cosh \rho \Gamma_0 + \dot{\rho} \Gamma_1) \Gamma_3 \Gamma_{01234} + \frac{1}{2} \dot{t} \sinh \rho (\dot{t} \cosh \rho \Gamma_0 + \dot{\rho} \Gamma_1) \Gamma_{01}.
\end{aligned} \tag{B.11}$$

Again, it is useful to perform a Lorentz rotation

$$\Gamma_0(s) = \Gamma_0 \cosh s + \Gamma_1 \sinh s, \quad \Gamma_1(s) = \Gamma_1 \cosh s + \Gamma_0 \sinh s. \tag{B.12}$$

with  $\sinh s = -\frac{\dot{\rho}}{\sqrt{\dot{t}^2 \cosh^2 \rho - \dot{\rho}^2}}$ ,  $\cosh s = \frac{\dot{t} \cosh \rho}{\sqrt{\dot{t}^2 \cosh^2 \rho - \dot{\rho}^2}}$ , Finally we get

$$D'_F = m \sinh \rho (\Gamma_0 \partial_\tau - \Gamma_3 \partial_\sigma) + \frac{m}{2} \dot{\rho} \cosh \rho \Gamma_0 + m^2 \sinh^2 \rho \Gamma_{124}. \tag{B.13}$$

Rescaling the fermions by  $\frac{1}{\sqrt{m \sinh \rho}}$  we end up with

$$D_F = \Gamma_0 \partial_\tau - \Gamma_3 \partial_\sigma + m \sinh \rho \Gamma_{124}. \tag{B.14}$$

## C Fluctuation Lagrangian for folded spinning string in $\mathbb{R} \times S^2$

Starting with the metric

$$\begin{aligned}
ds^2 &= -\left(\frac{1 + \frac{1}{4}\eta^2}{1 - \frac{1}{4}\eta^2}\right)^2 dt^2 + \frac{d\eta_k d\eta_k}{(1 - \frac{1}{4}\eta^2)^2} \\
&\quad + d\psi_1^2 + \cos^2 \psi_1 [d\psi_2^2 + \cos^2 \psi_2 (d\psi_3^2 + \cos^2 \psi_3 (d\psi_4^2 + \sin^2 \psi_4 d\phi^2))]
\end{aligned} \tag{C.1}$$

the folded spinning string on  $S^2$  of  $S^5$  is

$$t = \kappa \tau, \quad \eta_k = 0, \quad \psi_i = 0, \quad \psi_4 = \theta(\sigma), \quad \phi = w \tau \tag{C.2}$$

where  $k = 1, 2, 3, 4$  and  $i = 1, 2, 3$ . Fixing the static gauge on fluctuations by setting  $\tilde{t}$  and  $\tilde{\psi}_4$  to zero and expanding the Nambu action we obtain

$$\begin{aligned}
\tilde{S} &= \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[ (\partial_\sigma \tilde{\eta}_k)^2 - (\partial_\tau \tilde{\eta}_k)^2 + \kappa^2 \tilde{\eta}_k^2 + (\partial_\sigma \tilde{\psi}_i)^2 - (\partial_\tau \tilde{\psi}_i)^2 + (2w^2 \sin^2 \theta - \kappa^2) \tilde{\psi}_i^2 \right. \\
&\quad \left. + \frac{\kappa^2 \sin^2 \theta}{\theta'^2} [(\partial_\sigma \tilde{\phi})^2 - (\partial_\tau \tilde{\phi})^2] \right]
\end{aligned} \tag{C.3}$$

Setting  $f = \frac{\kappa \sin \theta}{\theta'} \tilde{\phi}$  we finally obtain

$$\begin{aligned}
\tilde{S} &= \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[ (\partial_\sigma \tilde{\eta}_k)^2 - (\partial_\tau \tilde{\eta}_k)^2 + \kappa^2 \tilde{\eta}_k^2 + (\partial_\sigma \tilde{\psi}_i)^2 - (\partial_\tau \tilde{\psi}_i)^2 + (2w^2 \sin^2 \theta - \kappa^2) \tilde{\psi}_i^2 \right. \\
&\quad \left. + (\partial_\sigma f)^2 - (\partial_\tau f)^2 + f^2 \kappa^2 \left(1 - \frac{2(\kappa^2 - w^2)}{\theta'^2}\right) \right]
\end{aligned} \tag{C.4}$$

To find the fermionic Lagrangian we start with  $\mathcal{L}_F = -2i\bar{\vartheta} D_F \vartheta$ . where  $\vartheta$  is the Majorana-Weyl 10d spinor,  $\bar{\vartheta} = \vartheta^t \Gamma_0$ ,  $\Gamma_{11}\vartheta = \vartheta$  and we shall use real Gamma matrices (as in, e.g., [30]). For more general 2-spin solution on  $S^3$  we find

$$D_F = s_1 \Gamma_0 \partial_0 - \Gamma_7 \partial_1 + u \Gamma_{078} \Gamma_{1234} + s_1 \frac{\kappa w_1 w_2}{2u^2} \Gamma_{789}, \quad (\text{C.5})$$

$$s_1 = \text{sign}(\theta'), \quad u = \sqrt{w_1^2 \cos^2 \theta + w_2^2 \sin^2 \theta}. \quad (\text{C.6})$$

In the case of  $w_1 = 0$ ,  $w_2 = w \neq 0$  this is

$$D_F = s_1 \Gamma_0 \partial_0 - \Gamma_7 \partial_1 + u \Gamma_{078} \Gamma_{1234}. \quad (\text{C.7})$$

The functional integral over the Majorana fermions gives the square root of the determinant of the operator  $\Gamma_0 D_F$ ,

$$\mathcal{D}_f = \det_{\text{Weyl}}^{1/2} (\Gamma_0 D_F). \quad (\text{C.8})$$

The operator  $\Gamma_0 D_F$  has a block structure respecting the Weyl condition  $[\Gamma_0 D_F, \Gamma_{11}] = 0$ . The spectrum is Weyl symmetric since for instance  $[\Gamma_0 D_F, \Gamma_6] = 0$  and  $\{\Gamma_{11}, \Gamma_6\} = 0$  and any eigenstate of  $\Gamma_0 D_F$  is mapped by  $\Gamma_6$  in an eigenstate with the same eigenvalue and opposite Weyl chirality. Thus we may relax the Weyl condition writing

$$\mathcal{D}_f = \det^{1/4} (\Gamma_0 D_F), \quad (\text{C.9})$$

where the determinant is defined on real 32 components spinors. Since  $\det \Gamma_7 = 1$  we have

$$\det (\Gamma_7 \Gamma_0 D_F) = \det (-s_1 \Gamma_7 \partial_0 + \Gamma_0 \partial_1 - u \Gamma_8 \Gamma_{1234}). \quad (\text{C.10})$$

Denoting by  $s_2$  the eigenvalue of  $\Gamma_{1234}$

$$\Gamma_{1234} \vartheta = s_2 \vartheta, \quad s_2 \in \{-1, 1\}, \quad (\text{C.11})$$

we can write

$$\mathcal{D}_f = \prod_{s_2=\pm 1} \det^{1/4} (-s_1 \Gamma_7 \partial_0 + \Gamma_0 \partial_1 - s_2 u \Gamma_8) = \prod_{s_2=\pm 1} \det^{1/8} (\partial_0^2 - \partial_1^2 + u^2 - s_2 u' \Gamma_{08}). \quad (\text{C.12})$$

Taking  $\Gamma_{08} = \sigma_3 \otimes \mathbb{I}_8$  which is possible on the space of definite  $\Gamma_{1234}$  chirality, we find

$$\mathcal{D}_f = \prod_{s=\pm 1} \det^2 (\partial_0^2 - \partial_1^2 + u^2 + s u'). \quad (\text{C.13})$$

The small string or small  $q$  expansion of the potential  $u$  ( $\kappa^2 = q + \frac{q^2}{2} + \dots$ , see section 6)

$$\begin{aligned} u &= \sin \sigma q^{1/2} + \left( \frac{\sin \sigma}{2} - \frac{\sin^3 \sigma}{4} \right) q^{3/2} + \left( \frac{\sin^5 \sigma}{16} - \frac{17 \sin^3 \sigma}{64} + \frac{11 \sin \sigma}{32} \right) q^{5/2} + \dots, \\ u' &= \cos \sigma q^{1/2} + \left( \frac{3 \cos^3 \sigma}{4} - \frac{\cos \sigma}{4} \right) q^{3/2} + \left( \frac{5 \cos^5 \sigma}{16} + \frac{11 \cos^3 \sigma}{64} - \frac{9 \cos \sigma}{64} \right) q^{5/2} + \dots, \\ u^2 &= \sin^2 \sigma q + \left( \sin^2 \sigma - \frac{\sin^4 \sigma}{2} \right) q^2 + \left( \frac{3 \sin^6 \sigma}{16} - \frac{25 \sin^4 \sigma}{32} + \frac{15 \sin^2 \sigma}{16} \right) q^3 + \dots. \end{aligned} \quad (\text{C.14})$$

Let mention also that the bosonic fluctuation masses in conformal gauge discussed in section 6 have the following expansions in the one-spin case when  $w_1 = 0$ ,  $Q_1 = 0$  ( $\kappa^2 = q + \frac{q^2}{2} + \dots$ , see section 6)

$$\begin{aligned}
M_X^2 &= (2 \sin^2 \sigma - 1) q + (-\sin^4 \sigma + 2 \sin^2 \sigma - \frac{1}{2}) q^2 + (\frac{3 \sin^6 \sigma}{8} - \frac{25 \sin^4 \sigma}{16} + \frac{15 \sin^2 \sigma}{8} - \frac{11}{32}) q^3 + \dots \\
M_\eta^2 &= -1 + (2 \sin^2 \sigma - \frac{1}{2}) q + (-\sin^4 \sigma + 2 \sin^2 \sigma - \frac{11}{32}) q^2 + (\frac{3 \sin^6 \sigma}{8} - \frac{25 \sin^4 \sigma}{16} + \frac{15 \sin^2 \sigma}{8} - \frac{17}{64}) q^3 + \dots \\
M_1^2 &= (2 \sin^2 \sigma - 1) q + (-\sin^4 \sigma + 2 \sin^2 \sigma - \frac{1}{2}) q^2 + (\frac{3 \sin^6 \sigma}{8} - \frac{25 \sin^4 \sigma}{16} + \frac{15 \sin^2 \sigma}{8} - \frac{11}{32}) q^3 + \dots, \\
M_2^2 &= -1 + (2 \sin^2 \sigma - \frac{3}{2}) q + (-\sin^4 \sigma + 2 \sin^2 \sigma - \frac{27}{32}) q^2 + (\frac{3 \sin^6 \sigma}{8} - \frac{25 \sin^4 \sigma}{16} + \frac{15 \sin^2 \sigma}{8} - \frac{39}{64}) q^3 + \dots \\
Q_2 &= -2 + (\sin^2 \sigma - \frac{1}{2}) q + (-\frac{1}{4} \sin^4 \sigma + \frac{3 \sin^2 \sigma}{4} - \frac{9}{32}) q^2 + (\frac{\sin^6 \sigma}{16} - \frac{11 \sin^4 \sigma}{32} + \frac{39 \sin^2 \sigma}{64} - \frac{25}{128}) q^3 + \dots
\end{aligned}$$

## D Perturbative computation of stability angles for pulsating string in $AdS_3$

Let us start with the bosonic type I fluctuations. Setting in Eq. (5.41)

$$\tau = \frac{2\mathbb{K}(\frac{R_+}{R_+ - R_-})}{\pi} \frac{1}{\sqrt{R_+ - R_-}} y, \quad 0 \leq y \leq 2\pi, \quad (\text{D.1})$$

and expanding in  $\kappa = \mathcal{E}_0 \rightarrow 0$ , we obtain

$$\mathcal{O}_I = \mathcal{O}_{I,0} + \mathcal{O}_{I,1} \kappa^2 + \dots, \quad \mathcal{O}_{I,0} = -\partial_y^2 - n^2, \quad \mathcal{O}_{I,1} = -\frac{3}{2} \partial_y^2 - z \sin^2 y, \quad \dots \quad (\text{D.2})$$

The evaluation of the stability angle for these operators is very simple and leads to the results in Eqs. (5.45).

For the bosonic type II fluctuations we get

$$\mathcal{O}_{II} = \mathcal{O}_{II,0} + \mathcal{O}_{II,1} \kappa^2 + \dots, \quad \mathcal{O}_{II,0} = -\partial_y^2 - n^2 + \frac{2}{\sin^2 y}, \quad \mathcal{O}_{II,1} = -\frac{3}{2} \partial_y^2 - 2 + \cos 2y + \frac{3}{\sin^2 y} \quad (\text{D.3})$$

The fluctuation equation for the  $\zeta$  field at leading order in  $\kappa \rightarrow 0$  limit is (for  $m = 1$ )

$$\left[ -\partial_y^2 - n^2 + \frac{2}{\sin^2 y} \right] \zeta_n^{(0)} = 0. \quad (\text{D.4})$$

We can look for a periodic solution such that  $\zeta_n^{(0)} \sin y \sim \tilde{\rho}$  is smooth. One finds one solution for  $n = 0, 1$  and two solutions for  $n \geq 2$ :

$$\zeta_0^{(0)} \sim \cot y, \quad \zeta_1^{(0)} \sim \csc y, \quad \zeta_{n \geq 2}^{(0) \pm} \sim \sqrt{\sin y} P_{n-\frac{1}{2}}^{\pm \frac{3}{2}}(\cos y), \quad (\text{D.5})$$

where  $P_n^m$  are associated Legendre polynomials. The first few cases for  $n \geq 2$  are

$$\zeta_2^{(0)+} \sim \csc y (\cos 3y - 3 \cos y), \quad \zeta_2^{(0)-} \sim \sin^2 y, \quad (\text{D.6})$$

$$\zeta_3^{(0)+} \sim \csc y (\cos 4y - 2 \cos 2y), \quad \zeta_3^{(0)-} \sim \sin^2 y \cos y, \quad (\text{D.7})$$

$$\zeta_4^{(0)+} \sim \csc y (3 \cos 5y - 5 \cos 3y), \quad \zeta_4^{(0)-} \sim \sin^2 y (2 + 3 \cos 2y). \quad (\text{D.8})$$



The general solution for  $\zeta^{(0)+}$  can be shown to be

$$\zeta_n^{(0)+} \sim \csc y \left[ \cos(n+1)y + \frac{n+1}{1-n} \cos(n-1)y \right], \quad (\text{D.9})$$

while the general solution for  $\zeta^{(0)-}$  is less explicit,

$$\zeta_n^{(0)-} \sim \sum_{\substack{0 \leq p \leq n \\ p-n \in 2\mathbb{Z}}} c_p \cos py, \quad (\text{D.10})$$

with certain coefficients  $c_p$ . The idea is now to do perturbation theory in  $\kappa$  hoping to find closely related stability angles for  $\zeta_n^{(0)\pm}$ . We can consider a perturbative expansion starting with the linear combination  $\zeta_n^{(0)+} + \mu_n \zeta_n^{(0)-}$ ,

$$\zeta_n = \zeta_n^{(0)+} + \mu_n \zeta_n^{(0)-} + \kappa^2 \zeta_n^{(1)} + \dots \quad (\text{D.11})$$

The mixing coefficient  $\mu_n$  is determined by the requirement that  $\zeta_n$  is quasiperiodic at order  $\kappa^2$ . In general, we find

$$\frac{\zeta_n(a+2\pi)}{\zeta_n(a)} = 1 + i\kappa^2 \nu_n^{(1)} + \dots \quad (\text{D.12})$$

We cannot compute  $\nu_n^{(1)}$  in a closed form as a function of  $n$  because we do not have an explicit expression for  $\zeta_n^{(0)-}$  as a closed function of  $n$ . Nevertheless, we can work out the procedure for several  $n$  and try a simple rational function of  $n$ . This works very well and the result for the expansion of the stability angle

$$\log \frac{\zeta_n(a+2\pi)}{\zeta_n(a)} = i\kappa^2 \nu_n^{(1)} + i\kappa^4 \nu_n^{(2)} + i\kappa^6 \nu_n^{(3)} + \dots \quad (\text{D.13})$$

agrees with the expressions in (5.52)

Let us mention that for the expansion of the fermionic operator  $\mathcal{O}_{III}$  from section 5.2 we get

$$\mathcal{O} = \mathcal{O}_0 + \mathcal{O}_{1/2} \mathcal{E} + \mathcal{O}_1 \mathcal{E}^2 + \dots, \quad (\text{D.14})$$

$$\mathcal{O}_0 = -\partial_y^2 - n^2, \quad \mathcal{O}_{1/2} = \pm i \cos y, \quad \mathcal{O}_1 = -\frac{3}{2} \partial_y^2 - \sin^2 y, \quad \dots \quad (\text{D.15})$$

## E On the expression for one-loop energy in terms of stability angles

Here we discuss at a level heuristic how one may obtain the semiclassical result (4.5) from one-loop effective action in the path integral approach.

Let us first consider the case of a stationary 2d soliton for which the fluctuation Lagrangian may have only  $\sigma$ -dependent coefficients. Then the 1-loop correction to the 2d energy can be found by computing the 1-loop *Euclidean* partition function<sup>26</sup>

$$E_{2d} = \frac{1}{2\mathcal{T}_\infty} \log \det \left[ -\partial_{\tau_e}^2 - \partial_\sigma^2 + V(\sigma) \right], \quad (\text{E.1})$$

---

<sup>26</sup>We consider the theory on a Euclidean cylinder  $R_{\tau_e} \times S_\sigma^1$ ,  $\tau_e = i\tau$ .

where  $\mathcal{T}_\infty \rightarrow \infty$  is an arbitrary time interval. Taking the trace over functions  $\sim e^{i\omega\tau_e}$ , we get

$$E_{2d} = \frac{1}{2\mathcal{T}_\infty} \times \mathcal{T}_\infty \times \int_{\mathbb{R}} \frac{d\omega}{2\pi} \log \det [\omega^2 - \partial_\sigma^2 + V(\sigma)] = \frac{1}{2} \int_{\mathbb{R}} \frac{d\omega}{2\pi} \sum_n \log (\omega^2 + \omega_n^2) , \quad (\text{E.2})$$

where the characteristic frequencies  $\omega_n$  are the eigenvalues of  $-\partial_\sigma^2 + V(\sigma)$ . In general, one has ( $R \rightarrow \infty$ )

$$\int_{-R}^R d\omega \log(\omega^2 + \omega_n^2) = -4R(1 - \log R) + 2\pi\omega_n + \mathcal{O}(R^{-1}) . \quad (\text{E.3})$$

Summing over bosons and fermions and ignoring the divergent terms (that will cancel in the present superstring case) we then get the familiar expression

$$E_{2d} = \frac{1}{2} \sum_n (-1)^F \omega_n . \quad (\text{E.4})$$

Let us also review a different representation for the determinant of the 1d operator like  $-\partial_\sigma^2 + V(\sigma)$  with periodic boundary conditions (see, e.g., section 4 of [6] for a summary). Using general notations, consider the problem

$$[-\partial_x^2 + V(x)] f(x) = \Lambda f(x) , \quad V(x+L) = V(x) . \quad (\text{E.5})$$

Its two independent solutions  $f_\pm(x) = e^{\pm i p(\Lambda) x} \chi_\pm(x)$ ,  $\chi_\pm(x+L) = \chi_\pm(x)$  satisfying

$$f_\pm(x+L) = e^{\pm i \nu} f_\pm(x) , \quad \nu = pL , \quad (\text{E.6})$$

define  $p$ , the “quasi-momentum”, and we also call  $\nu$  the “stability angle”. In general,  $p$  is a function of  $\Lambda, L$  and a functional of  $V$ . Then the determinant of the above operator computed with periodic boundary conditions on the eigen-functions can be represented as<sup>27</sup>

$$\log \det [-\partial_x^2 + V(x) - \Lambda] - \log \det [-\partial_x^2] = \ln [-4 \sin^2(\frac{1}{2}\nu)] . \quad (\text{E.7})$$

Let us now turn to the case of interest in this current paper: a time-dependent solution, periodic in real time. The idea is again to rotate to Euclidean time and take the infinite time interval limit and interpret the energy as a “ground-state” energy in the path integral context. Let us start with the logarithm of the real-time partition function on time interval  $r\mathcal{T}$  ( $r \rightarrow \infty$ ) where  $\mathcal{T}$  is the period,

$$\frac{1}{2} \log \det [\partial_\tau^2 - \partial_\sigma^2 + U(\tau)] = \frac{1}{2} \sum_{n=-\infty}^{\infty} \log \det [\partial_\tau^2 + n^2 + U(\tau)] . \quad (\text{E.8})$$

Here the sign of the potential  $U$  is chosen so that it is positive in the free massive particle case. We may then reduce the 2d determinant to a 1d one by using the Fourier transform ( $\partial_\sigma \rightarrow in$ ). Then  $E_{2d}$  may be defined by the Euclidean rotation of the above expression (E.8) divided by  $r\mathcal{T}$ . We may then use the above representation (E.7) for the 1d determinant in terms of the stability angle  $\nu$ . Noting that (i) going to Euclidean time suggests to set  $\nu \rightarrow i\nu$  and (ii) since we are on the interval  $r\mathcal{T}$  the accumulated stability angle will get a factor of  $r$ , we then finish with ( $\nu \rightarrow ir\nu$ )

$$E_{2d} = \lim_{r \rightarrow \infty} \frac{1}{2r\mathcal{T}} \sum_{n=-\infty}^{\infty} \log [4 \sinh^2 \frac{r\nu(n)}{2}] = \frac{1}{2\mathcal{T}} \sum_{n=-\infty}^{\infty} \nu(n) , \quad (\text{E.9})$$

where  $\nu$  is stability angle of the real-time problem on the period  $\mathcal{T}$  of the potential. This heuristic derivation reproduces the expression in (4.5).

<sup>27</sup>Here we included normalization to the free operator determinant that we will ignore in what follows (the corresponding constant factor will cancel in a superstring combination of determinants).

## F Comments on periodicity condition for fermions

As discussed at the end of section 7, in [27] it was suggested that for pulsating strings with odd winding number the fermions (defined using angular coordinates as tangent-space directions) should be chosen to be antiperiodic in  $\sigma$ . Below we shall comment on possible reason for that from flat space perspective and then present arguments against this interpretation in our curved space case by considering more general case with non-zero orbital momentum in  $S^5$ .

### F.1 Pulsating string solution in flat space

Let us start with pulsating solution in flat space. In cartesian coordinates

$$ds^2 = -dt^2 + dx^2 + dy^2 \quad (\text{F.1})$$

the pulsating solution is (this is of course the flat space limit of the  $S^2$  pulsating solution of section 2.1)

$$t = \kappa\tau, \quad x = \frac{\kappa}{m} \sin m\tau \cos m\sigma, \quad y = \frac{\kappa}{m} \sin m\tau \sin m\sigma \quad (\text{F.2})$$

The  $\theta^1 = \theta^2$   $\kappa$ -gauge fixed quadratic fermionic term in the GS action in flat space in cartesian coordinates is then (for simplicity in this section we define the fermionic Lagrangian without the overall factor of  $i$ )

$$L = 2\bar{\theta}D_F\theta, \quad D_F = -\rho_0\partial_0 + \rho_1\partial_1, \quad (\text{F.3})$$

$$\rho_0 = \kappa\Gamma_0 + \kappa \cos m\tau \cos m\sigma\Gamma_7 + \kappa \cos m\tau \sin m\sigma\Gamma_9, \quad \rho_1 = -\kappa \sin m\tau \sin m\sigma\Gamma_7 + \kappa \sin m\tau \cos m\sigma\Gamma_9$$

where we labelled the coordinate  $x$  as 7 and  $y$  as 9. We can get rid of the  $\sigma$  dependence in  $D_F$  by using the rotation

$$\theta = e^{-\frac{m\sigma}{2}\Gamma_7\Gamma_9}\tilde{\theta} \quad (\text{F.4})$$

leading to

$$\tilde{D}_F = -(\Gamma_0 + \cos m\tau\Gamma_7)\partial_0 + \Gamma_9 \sin m\tau\partial_1 + \frac{m}{2} \sin m\tau\Gamma_7 \quad (\text{F.5})$$

One can then put the fermionic Lagrangian in the standard free massless fermion form

$$\tilde{D}_F = -\Gamma_0\partial_0 + \Gamma_9\partial_1 \quad (\text{F.6})$$

using the redefinition (local boost and rescaling)

$$\tilde{\theta} = \sqrt{\cosh q} e^{-\frac{1}{2}q\Gamma_0\Gamma_7}\bar{\theta}, \quad \cosh q = \frac{1}{|\sin m\tau|} \quad (\text{F.7})$$

Note that the rotation (F.4) changes periodicity of the fermions: if we start with periodic  $\theta$  we get  $\tilde{\theta}$  (and thus also  $\bar{\theta}$ ) antiperiodic for odd  $m$ .

Let us now repeat the same computation starting with the same pulsating solution written in polar coordinates,

$$ds^2 = -dt^2 + d\psi^2 + \psi^2 d\phi^2, \quad t = \kappa\tau, \quad \psi = \frac{\kappa}{m} \sin m\tau, \quad \phi = m\sigma \quad (\text{F.8})$$

This is again the short string limit of the pulsating solution in  $S^2$  in (2.3). The bosonic part of the fluctuation Lagrangian is trivial, while the quadratic part of the GS superstring action written in general coordinates

$$L = (\sqrt{-g}g^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ})\bar{\theta}^I\rho_a D_b\theta^J. \quad (\text{F.9})$$

takes the following form in the  $\theta^1 = \theta^2$  gauge (here we use labels 0 for  $t$ , 7 for  $\psi$ , and 9 for  $\phi$ )<sup>28</sup>

$$L = 2\bar{\theta}D_F\theta, \quad D_F = -\rho_0 D_0 + \rho_1 D_1, \quad (\text{F.10})$$

$$D_F = -(\Gamma_0 + \cos m\tau\Gamma_7)\partial_0 + \Gamma_9 \sin m\tau\partial_1 + \frac{m}{2} \sin m\tau\Gamma_7 \quad (\text{F.11})$$

This operator is the same as in (F.5), so to put it in the standard form (F.6) one needs again the same local boost and rescaling as in (F.7). Since (F.7) does not change periodicity of the fermions, that seems to imply that to match the cartesian coordinate choice result, starting with GS action in coordinates (F.8) we need to assume that fermions are antiperiodic for odd  $m$ .

That conclusion may seem strange as we need standard periodic fermions to cancel corrections to ground-state energy. Also, it seems strange to assume that the choice of periodic/antiperiodic boundary conditions for the fermions in the original action (F.9) may depend on a specific choice of the bosonic solution: for any bosonic background the flat-space GS fermions are free (and periodic on a cylinder) in the light-cone gauge [36], so that periodicity/antiperiodicity issue is likely to be a gauge/coordinate artifact. Indeed, starting with the above solution in either cartesian or polar coordinates and writing the GS action (F.9) in the light-cone gauge  $\Gamma_+\theta^I = 0$ ,  $\Gamma_\pm = \frac{1}{2}(\mp\Gamma_0 + \Gamma_7)$  one ends up with the same free operator (F.6) defined for either periodic or antiperiodic fermions which appears to contradict the above conclusions.

Let us discuss explicitly what one finds in the light-cone gauge.

#### Light-cone gauge in polar coordinates

In polar coordinates we have

$$\begin{aligned} \rho_0 &= \kappa\Gamma_0 + \kappa \cos m\tau\Gamma_7, & \rho_1 &= \kappa \sin m\tau\Gamma_9, \\ D_0 &= \partial_0, & D_1 &= \partial_1 - \frac{m}{2}\Gamma_{79} \end{aligned} \quad (\text{F.12})$$

We fix the  $\kappa$  symmetry as

$$\Gamma_+\theta^I = 0, \quad \Gamma_\pm = \frac{1}{2}(\mp\Gamma_0 + \Gamma_7) \quad (\text{F.13})$$

The fermionic Lagrangian is then

$$L = -\bar{\theta}^1(1 + \cos m\tau)\Gamma_-(\partial_0 + \partial_1)\theta^1 - \bar{\theta}^2(1 + \cos m\tau)\Gamma_-(\partial_0 - \partial_1)\theta^2 + \frac{m}{2}\bar{\theta}^I \sin m\tau\Gamma_-\theta^I \quad (\text{F.14})$$

Performing the rescaling

$$\theta^I \rightarrow \frac{\tilde{\theta}^I}{\sqrt{1 + \cos m\tau}} \quad (\text{F.15})$$

we obtain

$$L = -\bar{\theta}^1\Gamma_-(\partial_0 + \partial_1)\theta^1 - \bar{\theta}^2\Gamma_-(\partial_0 - \partial_1)\theta^2 \quad (\text{F.16})$$

Thus starting with periodic fermions we get free action with periodic fermions.

#### Light-cone gauge with cartesian coordinates

In cartesian coordinates we have

$$\begin{aligned} \rho_0 &= \kappa\Gamma_0 + \kappa \cos m\tau \cos m\sigma\Gamma_7 + \kappa \cos m\tau \sin m\sigma\Gamma_9 \\ \rho_1 &= -\kappa \sin m\tau \sin m\sigma\Gamma_7 + \kappa \sin m\tau \cos m\sigma\Gamma_9, & D_0 &= \partial_0, & D_1 &= \partial_1 \end{aligned} \quad (\text{F.17})$$

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<sup>28</sup>One finds  $\rho_0 = \kappa\Gamma_0 + \kappa \cos m\tau\Gamma_7$ ,  $\rho_1 = \kappa \sin m\tau\Gamma_9$  and  $D_0 = \partial_0$ ,  $D_1 = \partial_1 - \frac{m}{2}\Gamma_{79}$ .

Fixing again the same light-cone gauge (F.13) we get the fermionic Lagrangian

$$\begin{aligned} L_F = & - \bar{\theta}^1 (1 + \cos m\tau \cos m\sigma + \sin m\tau \sin m\sigma) \Gamma_- (\partial_0 + \partial_1) \theta^1 \\ & - \bar{\theta}^2 (1 + \cos m\tau \cos m\sigma - \sin m\tau \sin m\sigma) \Gamma_- (\partial_0 - \partial_1) \theta^2 \end{aligned} \quad (\text{F.18})$$

Rescaling the fermions as

$$\theta^1 \rightarrow \frac{\tilde{\theta}^1}{\sqrt{1 + \cos m\tau \cos m\sigma + \sin m\tau \sin m\sigma}}, \quad \theta^2 \rightarrow \frac{\tilde{\theta}^2}{\sqrt{1 + \cos m\tau \cos m\sigma - \sin m\tau \sin m\sigma}} \quad (\text{F.19})$$

we obtain

$$L = -\bar{\theta}^1 \Gamma_- (\partial_0 + \partial_1) \theta^1 - \bar{\theta}^2 \Gamma_- (\partial_0 - \partial_1) \theta^2 \quad (\text{F.20})$$

Once again, starting with periodic fermions we end up with periodic fermions. This is in contrast to what happened in the  $\theta_1 = \theta_2$  gauge where a  $\sigma$ -dependent rotation was needed, which changed the periodicity of fermions.

## F.2 Pulsating string solution with extra angular momentum $J$ on $S^5$

To clarify what is going on further let us consider a generalization of the  $S^2$  pulsating solution to the presence of angular momentum  $J$  along a direction of  $S^5$  transverse to  $S^2$  [37, 11]. That will allow us to interpolate to large values of  $J$  and thus resolve the question about the fermionic boundary conditions by comparing to the BMN limit.

Starting with the metric of  $\mathbb{R} \times S^3$

$$ds^2 = -dt^2 + d\psi^2 + \sin^2 \psi d\phi^2 + \cos^2 \psi d\varphi^2 \quad (\text{F.21})$$

the solution with non-zero  $J = \sqrt{\lambda} \mathcal{J}$  is

$$t = \kappa\tau, \quad \psi = \psi(\tau), \quad \varphi = \varphi(\tau), \quad \phi = m\sigma, \quad (\text{F.22})$$

$$\dot{\psi}^2 + m^2 \sin^2 \psi + \frac{\mathcal{J}^2}{\cos^2 \psi} = \kappa^2, \quad \mathcal{J} = \cos^2 \psi \dot{\varphi} = \text{const}. \quad (\text{F.23})$$

Then in the quadratic part of the  $AdS_5 \times S^5$  GS Lagrangian

$$L = i(\eta^{ab} \delta^{IJ} - \epsilon^{ab} s^{IJ}) \bar{\theta}^I \rho_a \mathcal{D}_a \theta^J, \quad (\text{F.24})$$

$$\mathcal{D}_a \theta^I = D_a \theta^I - \frac{i}{2} \epsilon^{IJ} \Gamma_* \rho_a \theta^J, \quad s^{IJ} = (1, -1), \quad \Gamma_* = i\Gamma_{01234} \quad (\text{F.25})$$

we have

$$\rho_0 = \kappa\Gamma_0 + \dot{\psi}\Gamma_7 + \cos \psi \dot{\varphi} \Gamma_8, \quad \rho_1 = m \sin \psi \Gamma_9 \quad (\text{F.26})$$

$$D_0 = \partial_0 + \frac{1}{2} \sin \psi \dot{\varphi} \Gamma_{78}, \quad D_1 = \partial_1 - \frac{m}{2} \cos \psi \Gamma_{79} \quad (\text{F.27})$$

where we label the coordinates as 7 for  $\psi$ , 8 for  $\varphi$  and 9 for  $\phi$ . The resulting fermionic operator in the  $\theta^1 = \theta^2$  gauge is

$$\begin{aligned} D_F = & (\kappa\Gamma_0 + \dot{\psi}\Gamma_7 + \cos \psi \dot{\varphi} \Gamma_8) \partial_0 - m \sin \psi \Gamma_9 \partial_1 + \frac{1}{2} \sin \psi \dot{\varphi} (\kappa\Gamma_{078} + \dot{\psi}\Gamma_8 - \cos \psi \dot{\varphi} \Gamma_7) \\ & - \frac{m^2}{2} \sin \psi \cos \psi \Gamma_7 \pm m \sin \psi \Gamma_{09} (\dot{\psi}\Gamma_7 + \cos \psi \dot{\varphi} \Gamma_8) \end{aligned} \quad (\text{F.28})$$

where we projected onto eigenspaces with  $\Gamma_{1234} = \pm 1$ . Performing two boosts – in the (07) plane as

$$\theta = e^{-\frac{1}{2}\alpha\Gamma_0\Gamma_7}\tilde{\theta}, \quad \cosh \alpha = \frac{\kappa}{\sqrt{\kappa^2 - \dot{\psi}^2}} \quad (\text{F.29})$$

and in the (08) plane

$$\tilde{\theta} = e^{-\frac{1}{2}\beta\Gamma_0\Gamma_8}\bar{\theta}, \quad \cosh \beta = \frac{\sqrt{m^2 \sin^2 \psi + \cos^2 \psi \dot{\psi}^2}}{m \sin \psi} \quad (\text{F.30})$$

and rescaling

$$\bar{\theta} = \frac{1}{\sqrt{|m \sin \psi|}}\hat{\theta} \quad (\text{F.31})$$

we end up with  $D'_F = s\hat{D}_F$ ,  $s \equiv \text{sign}(\sin \psi)$ , where

$$\begin{aligned} \hat{D}_F &= \Gamma_0\partial_0 - \Gamma_9\partial_1 - \frac{\kappa m}{2} \frac{\dot{\psi} \cos(2\psi)}{m^2 \sin^2 \psi + \cos^2 \psi \dot{\psi}^2} \Gamma_{078} \\ &\pm \frac{m\dot{\psi} \sin \psi}{\sqrt{m^2 \sin^2 \psi + \cos^2 \psi \dot{\psi}^2}} \Gamma_{097} \pm \frac{\kappa\dot{\psi} \cos \psi}{\sqrt{m^2 \sin^2 \psi + \cos^2 \psi \dot{\psi}^2}} \Gamma_{098} \end{aligned} \quad (\text{F.32})$$

In the BMN limit of small string with large orbital momentum, i.e.  $m \rightarrow 0$ ,  $\psi \rightarrow 0$ ,  $\kappa = \mathcal{J}$ , we get the standard result

$$\hat{D}_F = \Gamma_0\partial_0 - \Gamma_9\partial_1 \pm \mathcal{J}\Gamma_{098}. \quad (\text{F.33})$$

In the limit of  $\mathcal{J} \rightarrow 0$  we end up with

$$\hat{D}_F = \Gamma_0\partial_0 - \Gamma_9\partial_1 \pm s\dot{\psi}\Gamma_{097} \quad (\text{F.34})$$

This is essentially the same as one finds by starting directly with  $J = 0$  as in Appendix A. Since the fermions must be periodic to match the BMN limit (F.33), they should also be periodic in the opposite pulsating string limit (F.34). While formally the transition between  $m = 0$  and  $m \neq 0$  cases may still look discontinuous, on physical grounds it seems natural to expect that near-BMN state represented by small pulsating string with large  $J$  should belong to a family of solutions that should all be quantized with periodic fermions.

Let us now consider the same computation choosing the light-cone gauge

$$\Gamma_+\theta^I = 0, \quad \Gamma_{\pm} \equiv \frac{1}{2}(\mp\Gamma_0 + \Gamma_8), \quad \Gamma_+\Gamma_- + \Gamma_-\Gamma_+ = 1 \quad (\text{F.35})$$

in the Lagrangian (F.24). Performing the rescaling

$$\theta^I = \frac{\tilde{\theta}^I}{\sqrt{\kappa + \dot{\psi} \cos \psi}} \quad (\text{F.36})$$

we obtain (omitting tilde on  $\theta$ )

$$\begin{aligned} L = & - \bar{\theta}^1 \Gamma_- (\partial_0 + \partial_1) \theta^1 - \bar{\theta}^2 \Gamma_- (\partial_0 - \partial_1) \theta^2 + \frac{m}{2} \frac{\kappa \cos \psi + \dot{\psi} \cos(2\psi)}{\kappa + \dot{\psi} \cos \psi} s^{IJ} \bar{\theta}^I \Gamma_- \Gamma_{79} \theta^J \\ & + 2\kappa \bar{\theta}^1 \Gamma_- \Pi \theta^2 - \frac{2m^2 \sin^2 \psi}{\kappa + \dot{\psi} \cos \psi} \bar{\theta}^1 \Gamma_- \Pi \theta^2 + \frac{2m\dot{\psi} \sin \psi}{\kappa + \dot{\psi} \cos \psi} \bar{\theta}^1 \Gamma_- \Gamma_{79} \Pi \theta^1 \end{aligned} \quad (\text{F.37})$$

where the second line comes from the second RR coupling term in (F.25) and  $\Pi = \Gamma_{1234}$ . For  $m \rightarrow 0$ ,  $\psi \rightarrow 0$ ,  $\kappa \rightarrow \mathcal{J}$  we get

$$L = -\bar{\theta}^1 \Gamma_- (\partial_0 + \partial_1) \theta^1 - \bar{\theta}^2 \Gamma_- (\partial_0 - \partial_1) \theta^2 + 2\mathcal{J} \bar{\theta}^1 \Gamma_- \Pi \theta^2 \quad (\text{F.38})$$

which is the familiar BMN expression. Note that for  $\psi \rightarrow 0$  but  $m$  arbitrary we get an additional  $m$ -dependent term. The limit  $\mathcal{J} \rightarrow 0$  is smooth and leads to

$$\begin{aligned} L = & -\bar{\theta}^1 \Gamma_- (\partial_0 + \partial_1) \theta^1 - \bar{\theta}^2 \Gamma_- (\partial_0 - \partial_1) \theta^2 + \frac{m}{2} \cos \psi s^{IJ} \bar{\theta}^I \Gamma_- \Gamma_{79} \theta^J + 2\kappa \bar{\theta}^1 \Gamma_- \Pi \theta^2 \\ & + \frac{2m\dot{\psi} \sin \psi}{\kappa} \bar{\theta}^1 \Gamma_- \Gamma_{79} \Pi \theta^2 - \frac{2m^2 \sin^2 \psi}{\kappa} \bar{\theta}^1 \Gamma_- \Pi \theta^2 \end{aligned} \quad (\text{F.39})$$

To conclude, embedding pulsating solution into a more general case with  $J \neq 0$  suggests that the fermion boundary conditions should be fixed universally rather than be sensitive to particular values of  $m$ .

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